CHAPTER XIV.

ON TRUE COMPOUND INTEREST AND THE LAW OF ORGANIC GROWTH.

LET there be a quantity growing in such a way that the increment of its growth, during a given time, shall always be proportional to its own magnitude. This resembles the process of reckoning interest on money at some fixed rate; for the bigger the capital, the bigger the amount of interest on it in a given time.

Now we must distinguish clearly between two cases, in our calculation, according as the calculation is made by what the arithmetic books call "simple interest," or by what they call "compound interest." For in the former case the capital remains fixed, while in the latter the interest is added to the capital, which therefore increases by successive additions.

(1) *At simple interest.* Consider a concrete case. Let the capital at start be £100, and let the rate of interest be 10 per cent. per annum. Then the increment to the owner of the capital will be £10 every year. Let him go on drawing his interest every year, and hoard it by putting it by in a

stocking, or locking it up in his safe. Then, if he goes on for 10 years, by the end of that time he will have received 10 increments of £10 each, or £100, making, with the original £100, a total of £200 in all. His property will have doubled itself in 10 years. If the rate of interest had been 5 per cent., he would have had to hoard for 20 years to double his property. If it had been only 2 per cent., he would have had to hoard for 50 years. It is easy to see that if the value of the yearly interest is $\frac{1}{n}$ of the capital, he must go on hoarding for n years in order to double his property.

Or, if y be the original capital, and the yearly interest is $\frac{y}{n}$, then, at the end of n years, his property will be

$$y + n\frac{y}{n} = 2y.$$

(2) *At compound interest.* As before, let the owner begin with a capital of £100, earning interest at the rate of 10 per cent. per annum; but, instead of hoarding the interest, let it be added to the capital each year, so that the capital grows year by year. Then, at the end of one year, the capital will have grown to £110; and in the second year (still at 10%) this will earn £11 interest. He will start the third year with £121, and the interest on that will be £12. 2*s.*; so that he starts the fourth year with £133. 2*s.*, and so on. It is easy to work it out, and find that at the end of the ten years the total capital

will have grown to £259. 7*s*. 6*d*. In fact, we see that at the end of each year, each pound will have earned $\frac{1}{10}$ of a pound, and therefore, if this is always added on, each year multiplies the capital by $\frac{11}{10}$; and if continued for ten years (which will multiply by this factor ten times over) the original capital will be multiplied by 2·59375. Let us put this into symbols. Put y_0 for the original capital; $\frac{1}{n}$ for the fraction added on at each of the n operations; and y_n for the value of the capital at the end of the n^{th} operation.

Then $$y_n = y_0\left(1+\frac{1}{n}\right)^n.$$

But this mode of reckoning compound interest once a year, is really not quite fair; for even during the first year the £100 ought to have been growing. At the end of half a year it ought to have been at least £105, and it certainly would have been fairer had the interest for the second half of the year been calculated on £105. This would be equivalent to calling it 5% per half-year; with 20 operations, therefore, at each of which the capital is multiplied by $\frac{21}{20}$. If reckoned this way, by the end of ten years the capital would have grown to £265. 8*s*.; for

$$(1+\tfrac{1}{20})^{20} = 2{\cdot}654.$$

But, even so, the process is still not quite fair; for, by the end of the first month, there will be some interest earned; and a half-yearly reckoning assumes that the capital remains stationary for six months at a time. Suppose we divided the year into 10 parts, and reckon a one-per-cent. interest for each tenth of the year. We now have 100 operations lasting over the ten years; or

$$y_n = £100\,(1 + \tfrac{1}{100})^{100};$$

which works out to £270. 8*s.*

Even this is not final. Let the ten years be divided into 1000 periods, each of $\frac{1}{100}$ of a year; the interest being $\frac{1}{10}$ per cent. for each such period; then

$$y_n = £100\,(1 + \tfrac{1}{1000})^{1000};$$

which works out to £271. 14*s.* $2\frac{1}{2}$*d.*

Go even more minutely, and divide the ten years into 10,000 parts, each $\frac{1}{1000}$ of a year, with interest at $\frac{1}{100}$ of 1 per cent. Then

$$y_n = £100\,(1 + \tfrac{1}{10{,}000})^{10{,}000};$$

which amounts to £271. 16*s.* 4*d.*

Finally, it will be seen that what we are trying to find is in reality the ultimate value of the expression $\left(1+\frac{1}{n}\right)^n$, which, as we see, is greater than 2; and which, as we take n larger and larger, grows closer and closer to a particular limiting value. However big you make n, the value of this expression grows nearer and nearer to the figure

$$2{\cdot}71828\ldots$$

a number *never to be forgotten.*

Let us take geometrical illustrations of these things. In Fig. 36, OP stands for the original value. OT is the whole time during which the value is growing. It is divided into 10 periods, in each of which there is an equal step up. Here $\frac{dy}{dx}$ is a constant; and if each step up is $\frac{1}{10}$ of the original OP, then, by 10 such steps, the height is doubled. If we had taken 20 steps,

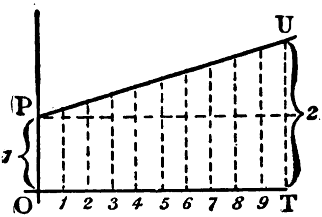


FIG. 36.

each of half the height shown, at the end the height would still be just doubled. Or n such steps, each of $\frac{1}{n}$ of the original height OP, would suffice to double the height. This is the case of simple interest. Here is 1 growing till it becomes 2.

In Fig. 37, we have the corresponding illustration of the geometrical progression. Each of the successive ordinates is to be $1+\frac{1}{n}$, that is, $\frac{n+1}{n}$ times as high as its predecessor. The steps up are not equal, because each step up is now $\frac{1}{n}$ of the ordinate *at that part* of the curve. If we had literally 10 steps, with $(1+\frac{1}{10})$ for the multiplying factor, the final total would be

$(1+\frac{1}{10})^{10}$ or 2·593 times the original 1. But if only we take n sufficiently large (and the corresponding $\frac{1}{n}$ sufficiently small), then the final value $\left(1+\frac{1}{n}\right)^n$ to which unity will grow will be 2·71828.

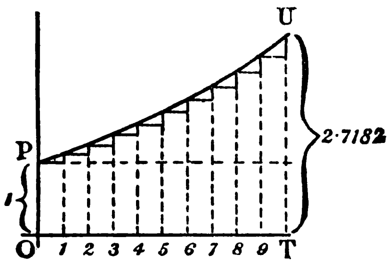


FIG. 37.

Epsilon. To this mysterious number 2·7182818 etc., the mathematicians have assigned as a symbol the Greek letter ϵ (pronounced *epsilon*) or the English letter *e*. All schoolboys know that the Greek letter π (called *pi*) stands for 3·141592 etc.; but how many of them know that *epsilon* means 2·71828? Yet it is an even more important number than π!

What, then, is *epsilon?*

Suppose we were to let 1 grow at simple interest till it became 2; then, if at the same nominal rate of interest, and for the same time, we were to let 1 grow at true compound interest, instead of simple, it would grow to the value *epsilon.*

This process of growing proportionately, at every instant, to the magnitude at that instant, some people call *a logarithmic rate* of growing. Unit logarithmic rate of growth is that rate which in unit time will cause 1 to grow to 2·718281. It might also be called the *organic rate* of growing: because it is characteristic of organic growth (in certain circumstances) that the increment of the organism in a given time is proportional to the magnitude of the organism itself.

If we take 100 per cent. as the unit of rate, and any fixed period as the unit of time, then the result of letting 1 grow *arithmetically* at unit rate, for unit time, will be 2, while the result of letting 1 grow *logarithmically* at unit rate, for the same time, will be 2·71828....

A little more about Epsilon. We have seen that we require to know what value is reached by the expression $\left(1+\frac{1}{n}\right)^n$, when n becomes indefinitely great. Arithmetically, here are tabulated a lot of values (which anybody can calculate out by the help of an ordinary table of logarithms) got by assuming $n=2$; $n=5$; $n=10$; and so on, up to $n=10{,}000$.

$$\begin{aligned}
(1+\tfrac{1}{2})^2 &= 2{\cdot}25.\\
(1+\tfrac{1}{5})^5 &= 2{\cdot}489.\\
(1+\tfrac{1}{10})^{10} &= 2{\cdot}594.\\
(1+\tfrac{1}{20})^{20} &= 2{\cdot}653.\\
(1+\tfrac{1}{100})^{100} &= 2{\cdot}704.\\
(1+\tfrac{1}{1000})^{1000} &= 2{\cdot}7171.\\
(1+\tfrac{1}{10{,}000})^{10{,}000} &= 2{\cdot}7182.
\end{aligned}$$

It is, however, worth while to find another way of calculating this immensely important figure.

Accordingly, we will avail ourselves of the binomial theorem, and expand the expression $\left(1+\frac{1}{n}\right)^n$ in that well-known way.

The binomial theorem gives the rule that

$$(a+b)^n = a^n + n\frac{a^{n-1}b}{\lfloor 1} + n(n-1)\frac{a^{n-2}b^2}{\lfloor 2} + n(n-1)(n-2)\frac{a^{n-3}b^3}{\lfloor 3} + \text{etc.}$$

Putting $a=1$ and $b=\frac{1}{n}$, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{\lfloor 2}\left(\frac{n-1}{n}\right) + \frac{1}{\lfloor 3}\frac{(n-1)(n-2)}{n^2} + \frac{1}{\lfloor 4}\frac{(n-1)(n-2)(n-3)}{n^3} + \text{etc.}$$

Now, if we suppose n to become indefinitely great, say a billion, or a billion billions, then $n-1$, $n-2$, and $n-3$, etc., will all be sensibly equal to n; and then the series becomes

$$\epsilon = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \text{etc.} \ldots$$

By taking this rapidly convergent series to as many terms as we please, we can work out the sum to any desired point of accuracy. Here is the working for ten terms:

	1·000000
dividing by 1	1·000000
dividing by 2	0·500000
dividing by 3	0·166667
dividing by 4	0·041667
dividing by 5	0·008333
dividing by 6	0·001389
dividing by 7	0·000198
dividing by 8	0·000025
dividing by 9	0·000002
Total	2·718281

ϵ is incommensurable with 1, and resembles π in being an interminable non-recurrent decimal.

The Exponential Series. We shall have need of yet another series.

Let us, again making use of the binomial theorem, expand the expression $\left(1+\frac{1}{n}\right)^{nx}$, which is the same as ϵ^x when we make n indefinitely great.

$$\epsilon^x = 1^{nx} + nx\frac{1^{nx-1}\left(\frac{1}{n}\right)}{\lfloor 1} + nx(nx-1)\frac{1^{nx-2}\left(\frac{1}{n}\right)^2}{\lfloor 2}$$

$$+nx(nx-1)(nx-2)\frac{1^{nx-3}\left(\frac{1}{n}\right)^3}{\lfloor 3} + \text{etc.}$$

$$= 1 \quad + x + \frac{1}{\lfloor 2} \cdot \frac{n^2x^2 - nx}{n^2}$$

$$+\frac{1}{\lfloor 3} \cdot \frac{n^3x^3 - 3n^2x^2 + 2nx}{n^3} + \text{etc.}$$

$$=1 \quad +x+\frac{x^2-\frac{x}{n}}{\lfloor 2}+\frac{x^3-\frac{3x^2}{n}+\frac{2x}{n^2}}{\lfloor 3}+\text{etc.}$$

But, when n is made indefinitely great, this simplifies down to the following:

$$\epsilon^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\text{etc.}\ldots$$

This series is called *the exponential series.*

The great reason why ϵ is regarded of importance is that ϵ^x possesses a property, not possessed by any other function of x, that *when you differentiate it its value remains unchanged*; or, in other words, its differential coefficient is the same as itself. This can be instantly seen by differentiating it with respect to x, thus:

$$\frac{d(\epsilon^x)}{dx}=0+1+\frac{2x}{1\cdot 2}+\frac{3x^2}{1\cdot 2\cdot 3}+\frac{4x^3}{1\cdot 2\cdot 3\cdot 4}$$
$$+\frac{5x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}$$

$$\text{or} \quad =1+x+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

which is exactly the same as the original series.

Now we might have gone to work the other way, and said: Go to; let us find a function of x, such that its differential coefficient is the same as itself. Or, is there any expression, involving only powers of x, which is unchanged by differentiation? Accordingly, let us *assume* as a general expression that

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.},$$

(in which the coefficients A, B, C, etc. will have to be determined), and differentiate it.

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{etc.}$$

Now, if this new expression is really to be the same as that from which it was derived, it is clear that A *must* $= B$; that $C = \frac{B}{2} = \frac{A}{1 \cdot 2}$; that $D = \frac{C}{3} = \frac{A}{1 \cdot 2 \cdot 3}$; that $E = \frac{D}{4} = \frac{A}{1 \cdot 2 \cdot 3 \cdot 4}$, etc.

The law of change is therefore that

$$y = A\left(1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}\right).$$

If, now, we take $A = 1$ for the sake of further simplicity, we have

$$y = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

Differentiating it any number of times will give always the same series over again.

If, now, we take the particular case of $A = 1$, and evaluate the series, we shall get simply

when $x = 1$, $y = 2{\cdot}718281$ etc.; that is, $y = \epsilon$;
when $x = 2$, $y = (2{\cdot}718281 \text{ etc})^2$; that is, $y = \epsilon^2$;
when $x = 3$, $y = (2{\cdot}718281 \text{ etc.})^3$; that is, $y = \epsilon^3$;

and therefore

when $x = x$, $y = (2{\cdot}718281 \text{ etc.})^x$; that is, $y = \epsilon^x$, thus finally demonstrating that

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

[NOTE.—*How to read exponentials.* For the benefit of those who have no tutor at hand it may be of use to state that ϵ^x is read as "*epsilon to the eksth power*;" or some people read it "*exponential eks.*" So ϵ^{pt} is read "*epsilon to the pee-teeth-power*" or "*exponential pee tee.*" Take some similar expressions:—Thus, ϵ^{-2} is read "*epsilon to the minus two power*" or "*exponential minus two.*" ϵ^{-ax} is read "*epsilon to the minus ay-eksth*" or "*exponential minus ay-eks.*"]

Of course it follows that ϵ^y remains unchanged if differentiated with respect to y. Also ϵ^{ax}, which is equal to $(\epsilon^a)^x$, will, when differentiated with respect to x, be $a\epsilon^{ax}$, because a is a constant.

Natural or Napierian Logarithms.

Another reason why ϵ is important is because it was made by Napier, the inventor of logarithms, the basis of his system. If y is the value of ϵ^x, then x is the *logarithm*, to the base ϵ, of y. Or, if

$$y = \epsilon^x,$$

then $$x = \log_\epsilon y.$$

The two curves plotted in Figs. 38 and 39 represent these equations.

The points calculated are:

For Fig. 38

x	0	0·5	1	1·5	2
y	1	1·65	2·71	4·50	7·69

For Fig. 39

y	1	2	3	4	8
x	0	0·69	1·10	1·39	2·08

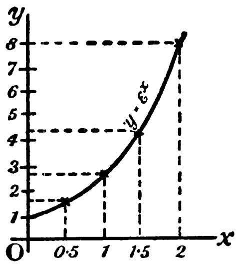


Fig. 38.

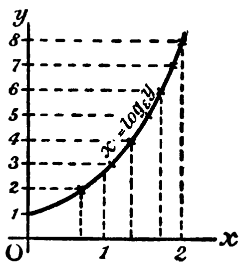


Fig. 39.

It will be seen that, though the calculations yield different points for plotting, yet the result is identical. The two equations really mean the same thing.

As many persons who use ordinary logarithms, which are calculated to base 10 instead of base ϵ, are unfamiliar with the "natural" logarithms, it may be worth while to say a word about them. The ordinary rule that adding logarithms gives the logarithm of the product still holds good; or

$$\log_\epsilon a + \log_\epsilon b = \log_\epsilon ab.$$

Also the rule of powers holds good;

$$n \times \log_\epsilon a = \log_\epsilon a^n.$$

But as 10 is no longer the basis, one cannot multiply by 100 or 1000 by merely adding 2 or 3 to the index. One can change the natural logarithm to the ordinary logarithm simply by multiplying it by 0·4343; or $\log_{10} x = 0{\cdot}4343 \times \log_\epsilon x$,

and conversely, $\log_\epsilon x = 2{\cdot}3026 \times \log_{10} x$.

A USEFUL TABLE OF "NAPIERIAN LOGARITHMS"

(Also called Natural Logarithms or Hyperbolic Logarithms).

Number	Log_ϵ	Number	Log_ϵ
1	0·0000	6	1·7918
1·1	0·0953	7	1·9459
1·2	0·1823	8	2·0794
1·5	0·4055	9	2·1972
1·7	0·5306	10	2·3026
2·0	0·6931	20	2·9957
2·2	0·7885	50	3·9120
2·5	0·9163	100	4·6052
2·7	0·9933	200	5·2983
2·8	1·0296	500	6·2146
3·0	1·0986	1,000	6·9078
3·5	1·2528	2,000	7·6010
4·0	1·3863	5.000	8·5172
4·5	1·5041	10,000	9·2104
5·0	1·6094	20,000	9·9035

Exponential and Logarithmic Equations.

Now let us try our hands at differentiating certain expressions that contain logarithms or exponentials.

Take the equation:

$$y = \log_\epsilon x.$$

First transform this into

$$\epsilon^y = x,$$

whence, since the differential of ϵ^y with regard to y is the original function unchanged (see p. 143),

$$\frac{dx}{dy} = \epsilon^y,$$

and, reverting from the inverse to the original function,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\epsilon^y} = \frac{1}{x}.$$

Now this is a very curious result. It may be written

$$\frac{d(\log_\epsilon x)}{dx} = x^{-1}.$$

Note that x^{-1} is a result that we could never have got by the rule for differentiating powers. That rule (page 25) is to multiply by the power, and reduce the power by 1. Thus, differentiating x^3 gave us $3x^2$; and differentiating x^2 gave $2x^1$. But differentiating x^0 gives us $0 \times x^{-1} = 0$, because x^0 is itself $= 1$, and is a constant. We shall have to come back to this curious fact that differentiating $\log_\epsilon x$ gives us $\frac{1}{x}$ when we reach the chapter on integrating.

Now, try to differentiate

$$y = \log_\epsilon(x+a),$$

that is

$$\epsilon^y = x + a;$$

we have $\frac{d(x+a)}{dy} = \epsilon^y$, since the differential of ϵ^y remains ϵ^y.

This gives $$\frac{dx}{dy}=e^y=x+a;$$

hence, reverting to the original function (see p. 131), we get

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{x+a}.$$

Next try $$y=\log_{10} x.$$

First change to natural logarithms by multiplying by the modulus 0·4343. This gives us

$$y=0{\cdot}4343\log_\epsilon x;$$

whence $$\frac{dy}{dx}=\frac{0{\cdot}4343}{x}.$$

The next thing is not quite so simple. **Try this:**

$$y=a^x.$$

Taking the logarithm of both sides, we get

$$\log_\epsilon y=x\log_\epsilon a,$$

or $$x=\frac{\log_\epsilon y}{\log_\epsilon a}=\frac{1}{\log_\epsilon a}\times\log_\epsilon y.$$

Since $\frac{1}{\log_\epsilon a}$ is a constant, we get

$$\frac{dx}{dy}=\frac{1}{\log_\epsilon a}\times\frac{1}{y}=\frac{1}{a^x\times\log_\epsilon a};$$

hence, reverting to the original function,

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=a^x\times\log_\epsilon a.$$

We see that, since

$$\frac{dx}{dy}\times\frac{dy}{dx}=1 \quad\text{and}\quad \frac{dx}{dy}=\frac{1}{y}\times\frac{1}{\log_\epsilon a},\quad \frac{1}{y}\times\frac{dy}{dx}=\log_\epsilon a.$$

We shall find that whenever we have an expression such as $\log_\epsilon y=$ a function of x, we always have $\frac{1}{y}\frac{dy}{dx}=$ the differential coefficient of the function cf x, so that we could have written at once, from $\log_\epsilon y=x\log_\epsilon a$,

$$\frac{1}{y}\frac{dy}{dx}=\log_\epsilon a \quad\text{and}\quad \frac{dy}{dx}=a^x\log_\epsilon a.$$

Let us now attempt further examples.

Examples.

(1) $y=\epsilon^{-ax}$. Let $-ax=z$; then $y=\epsilon^z$.

$$\frac{dy}{dz}=\epsilon^z;\ \frac{dz}{dx}=-a;\ \text{hence}\ \frac{dy}{dx}=-a\epsilon^{-ax}.$$

Or thus:

$$\log_\epsilon y=-ax;\ \frac{1}{y}\frac{dy}{dx}=-a;\ \frac{dy}{dx}=-ay=-a\epsilon^{-ax}.$$

(2) $y=\epsilon^{\frac{x^2}{3}}$. Let $\frac{x^2}{3}=z$; then $y=\epsilon^z$.

$$\frac{dy}{dz}=\epsilon^z;\ \frac{dz}{dx}=\frac{2x}{3};\ \frac{dy}{dx}=\frac{2x}{3}\epsilon^{\frac{x^2}{3}}.$$

Or thus:

$$\log_\epsilon y=\frac{x^2}{3};\ \frac{1}{y}\frac{dy}{dx}=\frac{2x}{3};\ \frac{dy}{dx}=\frac{2x}{3}\epsilon^{\frac{x^2}{3}}.$$

(3) $y=\epsilon^{\frac{2x}{x+1}}.$

$$\log_\epsilon y=\frac{2x}{x+1},\ \frac{1}{y}\frac{dy}{dx}=\frac{2(x+1)-2x}{(x+1)^2};$$

hence $$\frac{dy}{dx}=\frac{2}{(x+1)^2}\epsilon^{\frac{2x}{x+1}}.$$

Check by writing $\frac{2x}{x+1}=z.$

(4) $y=\epsilon^{\sqrt{x^2+a}}$. $\log_\epsilon y=(x^2+a)^{\frac{1}{2}}.$

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{(x^2+a)^{\frac{1}{2}}}\quad\text{and}\quad\frac{dy}{dx}=\frac{x\times\epsilon^{\sqrt{x^2+a}}}{(x^2+a)^{\frac{1}{2}}}.$$

(For if $(x^2+a)^{\frac{1}{2}}=u$ and $x^2+a=v$, $u=v^{\frac{1}{2}}$,

$$\frac{du}{dv}=\frac{1}{2v^{\frac{1}{2}}};\quad\frac{dv}{dx}=2x;\quad\frac{du}{dx}=\frac{x}{(x^2+a)^{\frac{1}{2}}}.)$$

Check by writing $\sqrt{x^2+a}=z.$

(5) $y=\log_\epsilon(a+x^3)$. Let $(a+x^3)=z$; then $y=\log_\epsilon z.$

$$\frac{dy}{dz}=\frac{1}{z};\quad\frac{dz}{dx}=3x^2;\quad\text{hence}\quad\frac{dy}{dx}=\frac{3x^2}{a+x^3}.$$

(6) $y=\log_\epsilon\{3x^2+\sqrt{a+x^2}\}$. Let $3x^2+\sqrt{a+x^2}=z$; then $y=\log_\epsilon z.$

$$\frac{dy}{dz}=\frac{1}{z};\quad\frac{dz}{dx}=6x+\frac{x}{\sqrt{x^2+a}};$$

$$\frac{dy}{dx}=\frac{6x+\frac{x}{\sqrt{x^2+a}}}{3x^2+\sqrt{a+x^2}}=\frac{x(1+6\sqrt{x^2+a})}{(3x^2+\sqrt{x^2+a})\sqrt{x^2+a}}.$$

(7) $y=(x+3)^2\sqrt{x-2}$.

$$\log_\epsilon y=2\log_\epsilon(x+3)+\tfrac{1}{2}\log_\epsilon(x-2).$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{(x+3)}+\frac{1}{2(x-2)};$$

$$\frac{dy}{dx}=(x+3)^2\sqrt{x-2}\left\{\frac{2}{x+3}+\frac{1}{2(x-2)}\right\}.$$

(8) $y=(x^2+3)^3(x^3-2)^{\frac{2}{3}}$.

$$\log_\epsilon y=3\log_\epsilon(x^2+3)+\tfrac{2}{3}\log_\epsilon(x^3-2);$$

$$\frac{1}{y}\frac{dy}{dx}=3\frac{2x}{x^2+3}+\frac{2}{3}\frac{3x^2}{x^3-2}=\frac{6x}{x^2+3}+\frac{2x^2}{x^3-2}.$$

(For if $u=\log_\epsilon(x^2+3)$, let $x^2+3=z$ and $u=\log_\epsilon z$.

$$\frac{du}{dz}=\frac{1}{z};\quad \frac{dz}{dx}=2x;\quad \frac{du}{dx}=\frac{2x}{x^2+3}.$$

Similarly, if $v=\log_\epsilon(x^3-2)$, $\dfrac{dv}{dx}=\dfrac{3x^2}{x^3-2}$) and

$$\frac{dy}{dx}=(x^2+3)^3(x^3-2)^{\frac{2}{3}}\left\{\frac{6x}{x^2+3}+\frac{2x^2}{x^3-2}\right\}.$$

(9) $y=\dfrac{\sqrt[2]{x^2+a}}{\sqrt[3]{x^3-a}}$.

$$\log_\epsilon y=\frac{1}{2}\log_\epsilon(x^2+a)-\frac{1}{3}\log_\epsilon(x^3-a).$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{2x}{x^2+a}-\frac{1}{3}\frac{3x^2}{x^3-a}=\frac{x}{x^2+a}-\frac{x^2}{x^3-a}$$

and $$\frac{dy}{dx}=\frac{\sqrt[2]{x^2+a}}{\sqrt[3]{x^3-a}}\left\{\frac{x}{x^2+a}-\frac{x^2}{x^3-a}\right\}.$$

(10) $y=\frac{1}{\log_\epsilon x}$.

$$\frac{dy}{dx}=\frac{\log_\epsilon x\times 0-1\times\frac{1}{x}}{\log_\epsilon^2 x}=-\frac{1}{x\log_\epsilon^2 x}.$$

(11) $y=\sqrt[3]{\log_\epsilon x}=(\log_\epsilon x)^{\frac{1}{3}}$. Let $z=\log_\epsilon x$; $y=z^{\frac{1}{3}}$.

$$\frac{dy}{dz}=\frac{1}{3}z^{-\frac{2}{3}};\quad \frac{dz}{dx}=\frac{1}{x};\quad \frac{dy}{dx}=\frac{1}{3x\sqrt[3]{\log_\epsilon^2 x}}.$$

(12) $y=\left(\frac{1}{a^x}\right)^{ax}$.

$$\log_\epsilon y=-ax\log_\epsilon a^x=-ax^2\,.\,\log_\epsilon a.$$

$$\frac{1}{y}\frac{dy}{dx}=-2ax\,.\,\log_\epsilon a$$

and $\frac{dy}{dx}=-2ax\left(\frac{1}{a^x}\right)^{ax}.\log_\epsilon a=-2xa^{1-ax^2}.\log_\epsilon a.$

Try now the following exercises.

Exercises XII. (See page 294 for Answers.)

(1) Differentiate $y=b(\epsilon^{ax}-\epsilon^{-ax})$.

(2) Find the differential coefficient with respect to t of the expression $u=at^2+2\log_\epsilon t$.

(3) If $y=n^t$, find $\frac{d(\log_\epsilon y)}{dt}$.

(4) Show that if $y=\frac{1}{b}\cdot\frac{a^{bx}}{\log_\epsilon a}$; $\frac{dy}{dx}=a^{bx}$.

(5) If $w=pv^n$, find $\frac{dw}{dv}$.

Differentiate

(6) $y=\log_\epsilon x^n$. (7) $y=3\epsilon^{-\frac{x}{x-1}}$.

(8) $y=(3x^2+1)\epsilon^{-5x}$. (9) $y=\log_\epsilon(x^a+a)$.

(10) $y=(3x^2-1)(\sqrt{x}+1)$.

(11) $y=\dfrac{\log_\epsilon(x+3)}{x+3}$. (12) $y=a^x\times x^a$.

(13) It was shown by Lord Kelvin that the speed of signalling through a submarine cable depends on the value of the ratio of the external diameter of the core to the diameter of the enclosed copper wire. If this ratio is called y, then the number of signals s that can be sent per minute can be expressed by the formula

$$s=ay^2\log_\epsilon\frac{1}{y};$$

where a is a constant depending on the length and the quality of the materials. Show that if these are given, s will be a maximum if $y=1\div\sqrt{\epsilon}$.

(14) Find the maximum or minimum of

$$y=x^3-\log_\epsilon x.$$

(15) Differentiate $y=\log_\epsilon(ax\epsilon^x)$.

(16) Differentiate $y=(\log_\epsilon ax)^3$.

The Logarithmic Curve.

Let us return to the curve which has its successive ordinates in geometrical progression, such as that represented by the equation $y=bp^x$.

We can see, by putting $x=0$, that b is the initial height of y.

Then when

$x=1,\ y=bp$; $x=2,\ y=bp^2$; $x=3,\ y=bp^3$, etc.

Also, we see that p is the numerical value of the ratio between the height of any ordinate and that of the next preceding it. In Fig. 40, we have taken p as $\frac{6}{5}$; each ordinate being $\frac{6}{5}$ as high as the preceding one.

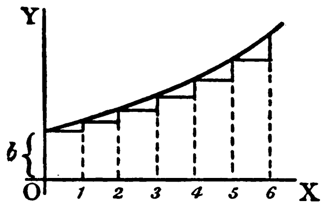


FIG. 40.

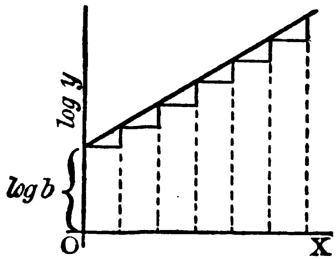


FIG. 41.

If two successive ordinates are related together thus in a constant ratio, their logarithms will have a constant difference; so that, if we should plot out a new curve, Fig. 41, with values of $\log_\epsilon y$ as ordinates, it would be a straight line sloping up by equal steps. In fact, it follows from the equation, that

$$\log_\epsilon y = \log_\epsilon b + x \cdot \log_\epsilon p,$$

whence $$\log_\epsilon y - \log_\epsilon b = x \cdot \log_\epsilon p.$$

Now, since $\log_\epsilon p$ is a mere number, and may be written as $\log_\epsilon p = a$, it follows that

$$\log_\epsilon \frac{y}{b} = ax,$$

and the equation takes the new form

$$y = b\epsilon^{ax}.$$

The Die-away Curve.

If we were to take p as a proper fraction (less than unity), the curve would obviously tend to sink downwards, as in Fig. 42, where each successive ordinate is $\frac{3}{4}$ of the height of the preceding one.

The equation is still

$$y = bp^x;$$

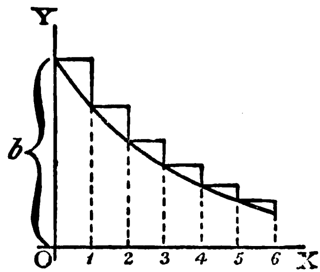


FIG. 42.

but since p is less than one, $\log_\epsilon p$ will be a negative quantity, and may be written $-a$; so that $p = \epsilon^{-a}$, and now our equation for the curve takes the form

$$y = b\epsilon^{-ax}.$$

The importance of this expression is that, in the case where the independent variable is *time*, the equation represents the course of a great many physical processes in which something is *gradually dying away*. Thus, the cooling of a hot body is represented (in Newton's celebrated "law of cooling") by the equation $\theta_t = \theta_0 \epsilon^{-at}$;

where θ_0 is the original excess of temperature of a hot body over that of its surroundings, θ_t the excess of temperature at the end of time t, and a is a constant—namely, the constant of decrement, depending on the amount of surface exposed by the body, and on its coefficients of conductivity and emissivity, etc.

A similar formula,

$$Q_t = Q_0 \epsilon^{-at},$$

is used to express the charge of an electrified body, originally having a charge Q_0, which is leaking away with a constant of decrement a; which constant depends in this case on the capacity of the body and on the resistance of the leakage-path.

Oscillations given to a flexible spring die out after a time; and the dying-out of the amplitude of the motion may be expressed in a similar way.

In fact ϵ^{-at} serves as a *die-away factor* for all those phenomena in which the rate of decrease is proportional to the magnitude of that which is decreasing; or where, in our usual symbols, $\frac{dy}{dt}$ is proportional at every moment to the value that y has at that moment. For we have only to inspect the curve, Fig. 42 above, to see that, at every part of it, the slope $\frac{dy}{dx}$ is proportional to the height y; the curve becoming flatter as y grows smaller. In symbols, thus

$$y = b\epsilon^{-ax}$$

or $$\log_\epsilon y = \log_\epsilon b - ax \log_\epsilon \epsilon = \log_\epsilon b - ax,$$

and, differentiating, $$\frac{1}{y}\frac{dy}{dx} = -a;$$

hence $$\frac{dy}{dx} = b\epsilon^{-ax} \times (-a) = -ay;$$

or, in words, the slope of the curve is downward, and proportional to y and to the constant a.

We should have got the same result if we had taken the equation in the form

$$y = bp^x;$$

for then $$\frac{dy}{dx} = bp^x \times \log_\epsilon p.$$

But $$\log_\epsilon p = -a;$$

giving us $$\frac{dy}{dx} = y \times (-a) = -ay,$$

as before.

The Time-constant. In the expression for the "die-away factor" ϵ^{-at}, the quantity a is the reciprocal of another quantity known as "*the time-constant,*" which we may denote by the symbol T. Then the die-away factor will be written $\epsilon^{-\frac{t}{T}}$; and it will be seen, by making $t = T$ that the meaning of T $\left(\text{or of } \frac{1}{a}\right)$ is that this is the length of time which it takes for the original quantity (called θ_0 or Q_0 in the preceding instances) to die away to $\frac{1}{\epsilon}$th part—that is to 0·3678—of its original value.

The values of ϵ^x and ϵ^{-x} are continually required in different branches of physics, and as they are given in very few sets of mathematical tables, some of the values are tabulated here for convenience.

x	ϵ^x	ϵ^{-x}	$1-\epsilon^{-x}$
0·00	1·0000	1·0000	0·0000
0·10	1·1052	0·9048	0·0952
0·20	1·2214	0·8187	0·1813
0·50	1·6487	0·6065	0·3935
0·75	2·1170	0·4724	0·5276
0·90	2·4596	0·4066	0·5934
1·00	2·7183	0·3679	0·6321
1·10	3·0042	0·3329	0·6671
1·20	3·3201	0·3012	0·6988
1·25	3·4903	0·2865	0·7135
1·50	4·4817	0·2231	0·7769
1·75	5·754	0·1738	0·8262
2·00	7·389	0·1353	0·8647
2·50	12·183	0·0821	0·9179
3·00	20·085	0·0498	0·9502
3·50	33·115	0·0302	0·9698
4·00	54·598	0·0183	0·9817
4·50	90·017	0·0111	0·9889
5·00	148·41	0·0067	0·9933
5·50	244·69	0·0041	0·9959
6·00	403·43	0·00248	0·99752
7·50	1808·04	0·00053	0·99947
10·00	22026·5	0·000045	0·999955

As an example of the use of this table, suppose there is a hot body cooling, and that at the beginning of the experiment (*i.e.* when $t=0$) it is 72° hotter than the surrounding objects, and if the time-constant of its cooling is 20 minutes (that is, if it takes 20 minutes for its excess of temperature to fall to $\frac{1}{\epsilon}$ part of 72°) then we can calculate to what it will have fallen in any given time t. For instance, let t be 60 minutes. Then $\frac{t}{T}=60\div 20=3$, and we shall have to find the value of ϵ^{-3}, and then multiply the original 72° by this. The table shows that ϵ^{-3} is 0·0498. So that at the end of 60 minutes the excess of temperature will have fallen to $72^\circ \times 0{\cdot}0498 = 3{\cdot}586^\circ$.

Further Examples.

(1) The strength of an electric current in a conductor at a time t secs. after the application of the electromotive force producing it is given by the expression $C=\frac{E}{R}\left\{1-\epsilon^{-\frac{Rt}{L}}\right\}$.

The time constant is $\frac{L}{R}$.

If $E=10$, $R=1$, $L=0{\cdot}01$; then when t is very large the term $1-\epsilon^{-\frac{Rt}{L}}$ becomes 1, and $C=\frac{E}{R}=10$; also

$$\frac{L}{R}=T=0{\cdot}01.$$

Its value at any time may be written:

$$C=10-10\epsilon^{-\frac{t}{0{\cdot}01}},$$

the time-constant being 0·01. This means that it takes 0·01 sec. for the variable term to fall to $\frac{1}{\epsilon}=0·3678$ of its initial value $10\epsilon^{-\frac{0}{0·01}}=10$.

To find the value of the current when $t=0·001$ sec., say, $\frac{t}{T}=0·1$, $\epsilon^{-0·1}=0·9048$ (from table).

It follows that, after 0·001 sec., the variable term is $0·9048\times10=9·048$, and the actual current is $10-9·048=0·952$.

Similarly, at the end of 0·1 sec.,

$$\frac{t}{T}=10\,;\ \epsilon^{-10}=0·000045\,;$$

the variable term is $10\times0·000045=0·00045$, the current being 9·9995.

(2) The intensity I of a beam of light which has passed through a thickness l cm. of some transparent medium is $I=I_0\epsilon^{-Kl}$, where I_0 is the initial intensity of the beam and K is a "constant of absorption."

This constant is usually found by experiments If it be found, for instance, that a beam of light has its intensity diminished by 18% in passing through 10 cms. of a certain transparent medium, this means that $82=100\times\epsilon^{-K\times10}$ or $\epsilon^{-10K}=0·82$, and from the table one sees that $10K=0·20$ very nearly; hence $K=0·02$.

To find the thickness that will reduce the intensity to half its value, one must find the value of l which satisfies the equality $50=100\times\epsilon^{-0·02l}$, or $0·5=\epsilon^{-0·02l}$.

It is found by putting this equation in its logarithmic form, namely,

$$\log 0{\cdot}5 = -0{\cdot}02 \times l \times \log \epsilon,$$

which gives

$$l = \frac{\bar{1}{\cdot}6990}{-0{\cdot}02 \times 0{\cdot}4343} = 34{\cdot}5 \text{ centimetres nearly.}$$

(3) The quantity Q of a radio-active substance which has not yet undergone transformation is known to be related to the initial quantity Q_0 of the substance by the relation $Q = Q_0 \epsilon^{-\lambda t}$, where λ is a constant and t the time in seconds elapsed since the transformation began.

For "Radium A," if time is expressed in seconds, experiment shows that $\lambda = 3{\cdot}85 \times 10^{-3}$. Find the time required for transforming half the substance. (This time is called the "mean life" of the substance.)

We have $$0{\cdot}5 = \epsilon^{-0{\cdot}00385t}.$$

$$\log 0{\cdot}5 = -0{\cdot}00385t \times \log \epsilon;$$

and $$t = 3 \text{ minutes very nearly.}$$

Exercises XIII. (See page 294 for Answers.)

(1) Draw the curve $y = b\epsilon^{-\frac{t}{T}}$; where $b = 12$, $T = 8$, and t is given various values from 0 to 20.

(2) If a hot body cools so that in 24 minutes its excess of temperature has fallen to half the initial amount, deduce the time-constant, and find how long it will be in cooling down to 1 per cent. of the original excess.

(3) Plot the curve $y=100(1-\epsilon^{-2t})$.

(4) The following equations give very similar curves:

$$\text{(i)}\quad y=\frac{ax}{x+b};$$

$$\text{(ii)}\quad y=a(1-\epsilon^{-\frac{x}{b}});$$

$$\text{(iii)}\quad y=\frac{a}{90^\circ}\operatorname{arc\,tan}\left(\frac{x}{b}\right).$$

Draw all three curves, taking $a=100$ millimetres; $b=30$ millimetres.

(5) Find the differential coefficient of y with respect to x, if (a) $y=x^x$; (b) $y=(\epsilon^x)^x$; (c) $y=\epsilon^{x^x}$.

(6) For "Thorium A," the value of λ is 5; find the "mean life," that is, the time taken by the transformation of a quantity Q of "Thorium A" equal to half the initial quantity Q_0 in the expression

$$Q=Q_0\epsilon^{-\lambda t};$$

t being in seconds.

(7) A condenser of capacity $K=4\times10^{-6}$, charged to a potential $V_0=20$, is discharging through a resistance of 10,000 ohms. Find the potential V after (a) 0·1 second; (b) 0·01 second; assuming that the fall of potential follows the rule $V=V_0\epsilon^{-\frac{t}{KR}}$.

(8) The charge Q of an electrified insulated metal sphere is reduced from 20 to 16 units in 10 minutes. Find the coefficient μ of leakage, if $Q=Q_0\times\epsilon^{-\mu t}$; Q_0 being the initial charge and t being in seconds. Hence find the time taken by half the charge to leak away.

(9) The damping on a telephone line can be ascertained from the relation $i=i_0\epsilon^{-\beta l}$, where i is the strength, after t seconds, of a telephonic current of initial strength i_0; l is the length of the line in kilometres, and β is a constant. For the Franco-English submarine cable laid in 1910, $\beta=0{\cdot}0114$. Find the damping at the end of the cable (40 kilometres), and the length along which i is still 8 % of the original current (limiting value of very good audition).

(10) The pressure p of the atmosphere at an altitude h kilometres is given by $p=p_0\epsilon^{-kh}$; p_0 being the pressure at sea-level (760 millimetres).

The pressures at 10, 20 and 50 kilometres being 199·2, 42·2, 0·32 millimetres respectively, find k in each case. Using the mean value of k, find the percentage error in each case.

(11) Find the minimum or maximum of $y=x^x$.

(12) Find the minimum or maximum of $y=x^{\frac{1}{x}}$.

(13) Find the minimum or maximum of $y=xa^{\frac{1}{x}}$.

CHAPTER XV.

HOW TO DEAL WITH SINES AND COSINES.

GREEK letters being usual to denote angles, we will take as the usual letter for any variable angle the letter θ ("theta").

Let us consider the function

$$y = \sin\theta.$$

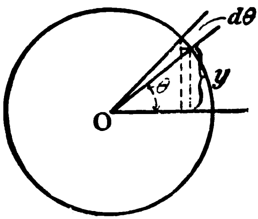


FIG. 43.

What we have to investigate is the value of $\dfrac{d(\sin\theta)}{d\theta}$; or, in other words, if the angle θ varies, we have to find the relation between the increment of the sine and the increment of the angle, both increments being indefinitely small in themselves. Examine Fig. 43, wherein, if the radius of the circle is unity, the height of y is the sine, and θ is the angle. Now, if θ is supposed to increase by the addition to it of the small angle $d\theta$—an element of angle—the height of y, the sine, will be increased by a small element dy. The new height $y+dy$ will be the sine of the new angle $\theta+d\theta$, or, stating it as an equation,

$$y+dy=\sin(\theta+d\theta);$$

and subtracting from this the first equation gives

$$dy=\sin(\theta+d\theta)-\sin\theta.$$

The quantity on the right-hand side is the difference between two sines, and books on trigonometry tell us how to work this out. For they tell us that if M and N are two different angles,

$$\sin M-\sin N=2\cos\frac{M+N}{2}\cdot\sin\frac{M-N}{2}.$$

If, then, we put $M=\theta+d\theta$ for one angle, and $N=\theta$ for the other, we may write

$$dy=2\cos\frac{\theta+d\theta+\theta}{2}\cdot\sin\frac{\theta+d\theta-\theta}{2},$$

or, $$dy=2\cos(\theta+\tfrac{1}{2}d\theta)\cdot\sin\tfrac{1}{2}d\theta.$$

But if we regard $d\theta$ as indefinitely small, then in the limit we may neglect $\frac{1}{2}d\theta$ by comparison with θ, and may also take $\sin\frac{1}{2}d\theta$ as being the same as $\frac{1}{2}d\theta$. The equation then becomes:

$$dy=2\cos\theta\times\tfrac{1}{2}d\theta;$$

$$dy=\cos\theta\cdot d\theta,$$

and, finally, $$\frac{dy}{d\theta}=\cos\theta.$$

The accompanying curves, Figs. 44 and 45, show, plotted to scale, the values of $y=\sin\theta$, and $\frac{dy}{d\theta}=\cos\theta$, for the corresponding values of θ.

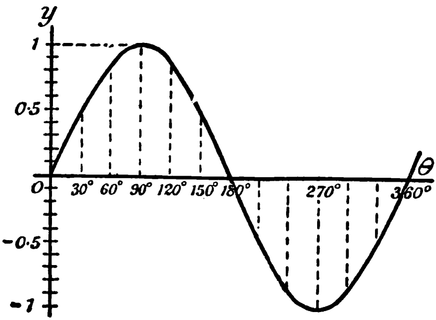


FIG. 44.

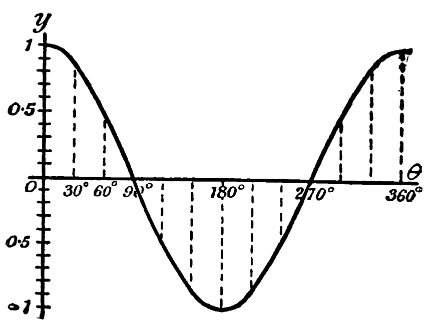


FIG. 45.

Take next the cosine.

Let $y=\cos\theta.$

Now $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right).$

Therefore

$$dy=d\left(\sin\left(\frac{\pi}{2}-\theta\right)\right)=\cos\left(\frac{\pi}{2}-\theta\right)\times d(-\theta),$$

$$=\cos\left(\frac{\pi}{2}-\theta\right)\times(-d\theta),$$

$$\frac{dy}{d\theta}=-\cos\left(\frac{\pi}{2}-\theta\right).$$

And it follows that

$$\frac{dy}{d\theta}=-\sin\theta.$$

Lastly, take the tangent.

Let $$y=\tan\theta,$$

$$=\frac{\sin\theta}{\cos\theta}.$$

The differential coefficient of $\sin\theta$ is $\frac{d(\sin\theta)}{d\theta}$, and the differential coefficient of $\cos\theta$ is $\frac{d(\cos\theta)}{d\theta}$. Applying the rule given on page 40 for differentiating a quotient of two functions, we get

$$\frac{dy}{d\theta}=\frac{\cos\theta\frac{d(\sin\theta)}{d\theta}-\sin\theta\frac{d(\cos\theta)}{d\theta}}{\cos^2\theta}$$

$$=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1}{\cos^2\theta},$$

or $$\frac{dy}{d\theta}=\sec^2\theta.$$

Collecting these results, we have:

y	$\frac{dy}{d\theta}$
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$

Sometimes, in mechanical and physical questions, as, for example, in simple harmonic motion and in wave-motions, we have to deal with angles that increase in proportion to the time. Thus, if T be the time of one complete *period*, or movement round the circle, then, since the angle all round the circle is 2π radians, or 360°, the amount of angle moved through in time t, will be

$$\theta=2\pi\frac{t}{T}, \text{ in radians,}$$

or $$\theta=360\frac{t}{T}, \text{ in degrees.}$$

If the *frequency*, or number of periods per second, be denoted by n, then $n=\frac{1}{T}$, and we may then write:

$$\theta=2\pi nt.$$

Then we shall have

$$y=\sin 2\pi nt.$$

If, now, we wish to know how the sine varies with respect to time, we must differentiate with respect, not to θ, but to t. For this we must resort to the artifice explained in Chapter IX., p. 67, and put

$$\frac{dy}{dt}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dt}.$$

Now $\frac{d\theta}{dt}$ will obviously be $2\pi n$; so that

$$\frac{dy}{dt}=\cos\theta\times 2\pi n$$

$$=2\pi n\cdot\cos 2\pi nt.$$

Similarly, it follows that

$$\frac{d(\cos 2\pi nt)}{dt}=-2\pi n\cdot\sin 2\pi nt.$$

Second Differential Coefficient of Sine or Cosine.

We have seen that when $\sin\theta$ is differentiated with respect to θ it becomes $\cos\theta$; and that when $\cos\theta$ is differentiated with respect to θ it becomes $-\sin\theta$; or, in symbols,

$$\frac{d^2(\sin\theta)}{d\theta^2}=-\sin\theta.$$

So we have this curious result that we have found a function such that if we differentiate it twice over, we get the same thing from which we started, but with the sign changed from + to −.

The same thing is true for the cosine; for differentiating $\cos\theta$ gives us $-\sin\theta$, and differentiating $-\sin\theta$ gives us $-\cos\theta$; or thus:

$$\frac{d^2(\cos\theta)}{d\theta^2} = -\cos\theta.$$

Sines and cosines are the only functions of which the second differential coefficient is equal and of opposite sign to the original function.

Examples.

With what we have so far learned we can now differentiate expressions of a more complex nature.

(1) $y = \text{arc}\sin x$.

If y is the arc whose sine is x, then $x = \sin y$.

$$\frac{dx}{dy} = \cos y.$$

Passing now from the inverse function to the original one, we get
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}.$$

Now
$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2};$$

hence
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}},$$

a rather unexpected result.

(2) $y=\cos^3\theta$.

This is the same thing as $y=(\cos\theta)^3$.

Let $\cos\theta=v$; then $y=v^3$; $\frac{dy}{dv}=3v^2$.

$$\frac{dv}{d\theta}=-\sin\theta.$$

$$\frac{dy}{d\theta}=\frac{dy}{dv}\times\frac{dv}{d\theta}=-3\cos^2\theta\sin\theta.$$

(3) $y=\sin(x+a)$.

Let $x+a=v$; then $y=\sin v$.

$$\frac{dy}{dv}=\cos v;\quad \frac{dv}{dx}=1\quad\text{and}\quad \frac{dy}{dx}=\cos(x+a).$$

(4) $y=\log_\epsilon\sin\theta$.

Let $\sin\theta=v$; $y=\log_\epsilon v$.

$$\frac{dy}{dv}=\frac{1}{v};\quad \frac{dv}{d\theta}=\cos\theta;$$

$$\frac{dy}{d\theta}=\frac{1}{\sin\theta}\times\cos\theta=\cot\theta.$$

(5) $y=\cot\theta=\dfrac{\cos\theta}{\sin\theta}$.

$$\frac{dy}{d\theta}=\frac{-\sin^2\theta-\cos^2\theta}{\sin^2\theta}$$

$$=-(1+\cot^2\theta)=-\operatorname{cosec}^2\theta.$$

(6) $y=\tan 3\theta$.

Let $3\theta=v$; $y=\tan v$; $\frac{dy}{dv}=\sec^2 v$.

$$\frac{dv}{d\theta}=3;\quad \frac{dy}{d\theta}=3\sec^2 3\theta.$$

(7) $y=\sqrt{1+3\tan^2\theta}$; $y=(1+3\tan^2\theta)^{\frac{1}{2}}$.

Let $3\tan^2\theta=v$.

$$y=(1+v)^{\frac{1}{2}};\quad \frac{dy}{dv}=\frac{1}{2\sqrt{1+v}}$$ (see p. 68);

$$\frac{dv}{d\theta}=6\tan\theta\sec^2\theta$$

(for, if $\tan\theta=u$,

$$v=3u^2;\quad \frac{dv}{du}=6u;\quad \frac{du}{d\theta}=\sec^2\theta;$$

hence $$\frac{dv}{d\theta}=6\tan\theta\sec^2\theta);$$

hence $$\frac{dy}{d\theta}=\frac{6\tan\theta\sec^2\theta}{2\sqrt{1+3\tan^2\theta}}.$$

(8) $y=\sin x\cos x$.

$$\frac{dy}{dx}=\sin x(-\sin x)+\cos x\times\cos x$$
$$=\cos^2 x-\sin^2 x.$$

Exercises XIV. (See page 295 for Answers.)

(1) Differentiate the following:

(i) $y=A\sin\left(\theta-\frac{\pi}{2}\right)$.

(ii) $y=\sin^2\theta$; and $y=\sin 2\theta$.

(iii) $y=\sin^3\theta$; and $y=\sin 3\theta$.

(2) Find the value of θ for which $\sin\theta\times\cos\theta$ is a maximum.

(3) Differentiate $y=\frac{1}{2\pi}\cos 2\pi nt$.

(4) If $y=\sin a^x$, find $\frac{dy}{dx}$.

(5) Differentiate $y=\log_\epsilon \cos x$.

(6) Differentiate $y=18{\cdot}2 \sin(x+26^\circ)$.

(7) Plot the curve $y=100\sin(\theta-15^\circ)$; and show that the slope of the curve at $\theta=75^\circ$ is half the maximum slope.

(8) If $y=\sin\theta\cdot\sin 2\theta$, find $\frac{dy}{d\theta}$.

(9) If $y=a\cdot\tan^m(\theta^n)$, find the differential coefficient of y with respect to θ.

(10) Differentiate $y=\epsilon^x \sin^2 x$.

(11) Differentiate the three equations of Exercises XIII. (p. 163), No. 4, and compare their differential coefficients, as to whether they are equal, or nearly equal, for very small values of x, or for very large values of x, or for values of x in the neighbourhood of $x=30$.

(12) Differentiate the following:

(i) $y=\sec x$.

(ii) $y=\text{arc}\cos x$.

(iii) $y=\text{arc}\tan x$.

(iv) $y=\text{arc}\sec x$.

(v) $y=\tan x\times\sqrt{3\sec x}$.

(13) Differentiate $y=\sin(2\theta+3)^{2{\cdot}3}$.

(14) Differentiate $y=\theta^3+3\sin(\theta+3)-3^{\sin\theta}-3^\theta$.

(15) Find the maximum or minimum of $y=\theta\cos\theta$.

CHAPTER XVI.

PARTIAL DIFFERENTIATION.

WE sometimes come across quantities that are functions of more than one independent variable. Thus, we may find a case where y depends on two other variable quantities, one of which we will call u and the other v. In symbols

$$y = f(u, v).$$

Take the simplest concrete case.

Let $$y = u \times v.$$

What are we to do? If we were to treat v as a constant, and differentiate with respect to u, we should get $$dy_v = v\,du;$$

or if we treat u as a constant, and differentiate with respect to v, we should have:

$$dy_u = u\,dv.$$

The little letters here put as subscripts are to show which quantity has been taken as constant in the operation.

Another way of indicating that the differentiation has been performed only *partially*, that is, has been performed only with respect to *one* of the independent variables, is to write the differential coefficients with Greek deltas, like ∂, instead of little d. In this way

$$\frac{\partial y}{\partial u}=v,$$

$$\frac{\partial y}{\partial v}=u.$$

If we put in these values for v and u respectively, we shall have

$$\left.\begin{aligned} dy_v &= \frac{\partial y}{\partial u}du, \\ dy_u &= \frac{\partial y}{\partial v}dv, \end{aligned}\right\} \text{ which are } \textit{partial differentials.}$$

But, if you think of it, you will observe that the total variation of y depends on *both* these things at the same time. That is to say, if both are varying, the real dy ought to be written

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv;$$

and this is called a *total differential.* In some books it is written $dy=\left(\frac{dy}{du}\right)du+\left(\frac{dy}{dv}\right)dv.$

Example (1). Find the partial differential coefficients of the expression $w=2ax^2+3bxy+4cy^3$. The answers are:

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= 4ax+3by. \\ \frac{\partial w}{\partial y} &= 3bx+12cy^2. \end{aligned}\right\}$$

The first is obtained by supposing y constant, the second is obtained by supposing x constant; then

$$dw=(4ax+3by)dx+(3bx+12cy^2)dy.$$

Example (2). Let $z=x^y$. Then, treating first y and then x as constant, we get in the usual way

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=yx^{y-1},\\ \frac{\partial z}{\partial y}&=x^y\times\log_\epsilon x,\end{aligned}\right\}$$

so that $dz=yx^{y-1}dx+x^y\log_\epsilon x\,dy$.

Example (3). A cone having height h and radius of base r, has volume $V=\frac{1}{3}\pi r^2h$. If its height remains constant, while r changes, the ratio of change of volume, with respect to radius, is different from ratio of change of volume with respect to height which would occur if the height were varied and the radius kept constant, for

$$\left.\begin{aligned}\frac{\partial V}{\partial r}&=\frac{2\pi}{3}rh,\\ \frac{\partial V}{\partial h}&=\frac{\pi}{3}r^2.\end{aligned}\right\}$$

The variation when both the radius and the height change is given by $dV=\frac{2\pi}{3}rh\,dr+\frac{\pi}{3}r^2dh$.

Example (4). In the following example F and f denote two arbitrary functions of any form whatsoever. For example, they may be sine-functions, or exponentials, or mere algebraic functions of the two independent variables, t and x. This being understood, let us take the expression

$$y = F(x+at)+f(x-at),$$

or,
$$y = F(w)+f(v);$$

where
$$w = x+at, \text{ and } v = x-at.$$

Then
$$\frac{dy}{dx} = \frac{\partial F(w)}{\partial w}\cdot\frac{dw}{dx}+\frac{\partial f(v)}{\partial v}\cdot\frac{dv}{dx}$$
$$= F'(w)\cdot 1+f'(v)\cdot 1$$

(where the figure 1 is simply the coefficient of x in w and v);

and
$$\frac{d^2y}{dx^2} = F''(w)+f''(v).$$

Also
$$\frac{dy}{dt} = \frac{\partial F(w)}{\partial w}\cdot\frac{dw}{dt}+\frac{\partial f(v)}{\partial v}\cdot\frac{dv}{dt}$$
$$= F'(w)\cdot a-f'(v)a;$$

and
$$\frac{d^2y}{dt^2} = F''(w)a^2+f''(v)a^2;$$

whence
$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}.$$

This differential equation is of immense importance in mathematical physics. (See also page 247.)

Maxima and Minima of Functions of two Independent Variables.

Example (5). Let us take up again Exercise IX., p. 110, No. 4.

Let x and y be the length of two of the portions of the string. The third is $30-(x+y)$, and the area of the triangle is $A=\sqrt{s(s-x)(s-y)(s-30+x+y)}$, where s is the half perimeter, 15, so that $A=\sqrt{15P}$, where

$$P=(15-x)(15-y)(x+y-15)$$
$$=xy^2+x^2y-15x^2-15y^2-45xy+450x+450y-3375.$$

Clearly A is maximum when P is maximum.

$$dP=\frac{\partial P}{\partial x}\,dx+\frac{\partial P}{\partial y}\,dy.$$

For a maximum (clearly it will not be a minimum in this case), one must have simultaneously

$$\frac{\partial P}{\partial x}=0 \text{ and } \frac{\partial P}{\partial y}=0;$$

that is, $$\left.\begin{aligned}2xy-30x+y^2-45y+450=0,\\ 2xy-30y+x^2-45x+450=0.\end{aligned}\right\}$$

An immediate solution is $x=y$.

If we now introduce this condition in the value of P, we find

$$P=(15-x)^2(2x-15)=2x^3-75x^2+900x-3375.$$

For maximum or minimum, $\frac{dP}{dx}=6x^2-150x+900=0$, which gives $x=15$ or $x=10$.

Clearly $x=15$ gives zero area; $x=10$ gives the maximum, for $\frac{d^2P}{dx^2}=12x-150$, which is $+30$ for $x=15$ and -30 for $x=10$.

Example (6). Find the dimensions of an ordinary railway coal truck with rectangular ends, so that, for a given volume V the area of sides and floor together is as small as possible.

The truck is a rectangular box open at the top. Let x be the length and y be the width; then the depth is $\frac{V}{xy}$. The surface area is $S = xy + \frac{2V}{x} + \frac{2V}{y}$.

$$dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial y}\,dy = \left(y - \frac{2V}{x^2}\right)dx + \left(x - \frac{2V}{y^2}\right)dy.$$

For minimum (clearly it won't be a maximum here),

$$y - \frac{2V}{x^2} = 0,\quad x - \frac{2V}{y^2} = 0.$$

Here also, an immediate solution is $x = y$, so that $S = x^2 + \frac{4V}{x}$, $\frac{dS}{dx} = 2x - \frac{4V}{x^2} = 0$ for minimum, and

$$x = \sqrt[3]{2V}.$$

Exercises XV. (See page 296 for Answers.)

(1) Differentiate the expression $\frac{x^3}{3} - 2x^3y - 2y^2x + \frac{y}{3}$ with respect to x alone, and with respect to y alone.

(2) Find the partial differential coefficients with respect to x, y and z, of the expression

$$x^2yz + xy^2z + xyz^2 + x^2y^2z^2.$$

(3) Let $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

Find the value of $\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} + \frac{\partial r}{\partial z}$. Also find the value of $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}$.

(4) Find the total differential of $y = u^v$.

(5) Find the total differential of $y=u^3 \sin v$; of $y=(\sin x)^u$; and of $y=\frac{\log_\epsilon u}{v}$.

(6) Verify that the sum of three quantities x, y, z, whose product is a constant k, is minimum when these three quantities are equal.

(7) Find the maximum or minimum of the function

$$u=x+2xy+y.$$

(8) The post-office regulations state that no parcel is to be of such a size that its length plus its girth exceeds 6 feet. What is the greatest volume that can be sent by post (a) in the case of a package of rectangular cross section; (b) in the case of a package of circular cross section.

(9) Divide π into 3 parts such that the continued product of their sines may be a maximum or minimum.

(10) Find the maximum or minimum of $u=\frac{\epsilon^{x+y}}{xy}$.

(11) Find maximum and minimum of

$$u=y+2x-2\log_\epsilon y-\log_\epsilon x.$$

(12) A telpherage bucket of given capacity has the shape of a horizontal isosceles triangular prism with the apex underneath, and the opposite face open. Find its dimensions in order that the least amount of iron sheet may be used in its construction.

CHAPTER XVII.

INTEGRATION.

The great secret has already been revealed that this mysterious symbol $\int$, which is after all only a long *S*, merely means "the sum of," or "the sum of all such quantities as." It therefore resembles that other symbol Σ (the Greek *Sigma*), which is also a sign of summation. There is this difference, however, in the practice of mathematical men as to the use of these signs, that while Σ is generally used to indicate the sum of a number of finite quantities, the integral sign $\int$ is generally used to indicate the summing up of a vast number of small quantities of indefinitely minute magnitude, mere elements in fact, that go to make up the total required. Thus $\int dy = y$, and $\int dx = x$.

Any one can understand how the whole of anything can be conceived of as made up of a lot of little bits; and the smaller the bits the more of them there will be. Thus, a line one inch long may be conceived as made up of 10 pieces, each $\frac{1}{10}$ of an inch long; or of 100 parts, each part being $\frac{1}{100}$ of an inch long;

or of 1,000,000 parts, each of which is $\frac{1}{1,000,000}$ of an inch long; or, pushing the thought to the limits of conceivability, it may be regarded as made up of an infinite number of elements each of which is infinitesimally small.

Yes, you will say, but what is the use of thinking of anything that way? Why not think of it straight off, as a whole? The simple reason is that there are a vast number of cases in which one cannot calculate the bigness of the thing as a whole without reckoning up the sum of a lot of small parts. The process of "*integrating*" is to enable us to calculate totals that otherwise we should be unable to estimate directly.

Let us first take one or two simple cases to familiarize ourselves with this notion of summing up a lot of separate parts.

Consider the series:

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64}+\text{etc.}$$

Here each member of the series is formed by taking it half the value of the preceding. What is the value of the total if we could go on to an infinite number of terms? Every schoolboy knows that the answer is 2. Think of it, if you like, as a line. Begin with

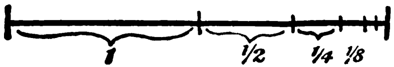


FIG 46.

one inch; add a half inch; add a quarter; add an eighth; and so on. If at any point of the operation we stop, there will still be a piece wanting to make up the whole 2 inches; and the piece wanting will always be the same size as the last piece added. Thus, if after having put together 1, $\frac{1}{2}$, and $\frac{1}{4}$, we stop, there will be $\frac{1}{4}$ wanting. If we go on till we have added $\frac{1}{64}$, there will still be $\frac{1}{64}$ wanting. The remainder needed will always be equal to the last term added. By an infinite number of operations only should we reach the actual 2 inches. Practically we should reach it when we got to pieces so small that they could not be drawn—that would be after about 10 terms, for the eleventh term is $\frac{1}{1024}$. If we want to go so far that not even a Whitworth's measuring machine would detect it, we should merely have to go to about 20 terms. A microscope would not show even the 18$^{\text{th}}$ term! So the infinite number of operations is no such dreadful thing after all. The *integral* is simply the whole lot. But, as we shall see, there are cases in which the integral calculus enables us to get at the *exact* total that there would be as the result of an infinite number of operations. In such cases the integral calculus gives us a *rapid* and easy way of getting at a result that would otherwise require an interminable lot of elaborate working out. So we had best lose no time in learning *how to integrate.*

Slopes of Curves, and the Curves themselves.

Let us make a little preliminary enquiry about the slopes of curves. For we have seen that differentiating a curve means finding an expression for its slope (or for its slopes at different points). Can we perform the reverse process of reconstructing the whole curve if the slope (or slopes) are prescribed for us?

Go back to case (2) on p. 84. Here we have the simplest of curves, a sloping line with the equation

$$y = ax + b.$$

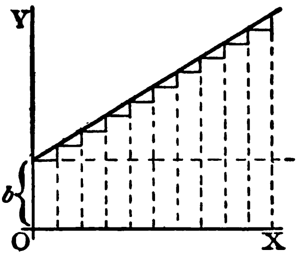
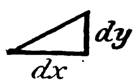


FIG. 47

We know that here b represents the initial height of y when $x=0$, and that a, which is the same as $\frac{dy}{dx}$, is the "slope" of the line. The line has a constant slope. All along it the elementary triangles 

have the same proportion between height and base. Suppose we were to take the dx's and dy's of finite magnitude, so that 10 dx's made up one inch, then there would be ten little triangles like

Now, suppose that we were ordered to reconstruct the "curve," starting merely from the information that $\frac{dy}{dx} = a$. What could we do? Still taking the little d's as of finite size, we could draw 10 of them, all with the same slope, and then put them together, end to end, like this:

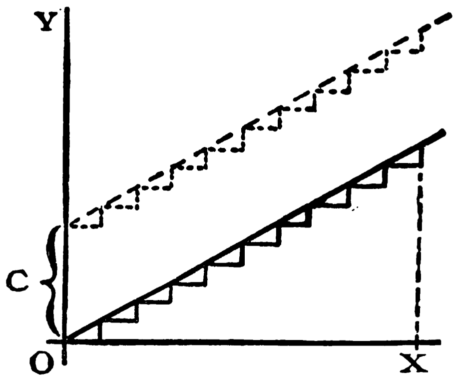


FIG. 48.

And, as the slope is the same for all, they would join to make, as in Fig. 48, a sloping line sloping with the correct slope $\frac{dy}{dx} = a$. And whether we take the dy's and dx's as finite or infinitely small, as they are all alike, clearly $\frac{y}{x}=a$, if we reckon y as the total of all the dy's, and x as the total of all the dx's. But whereabouts are we to put this sloping line? Are we to start at the origin O, or higher up? As the only information we have is as to the slope, we are without any instructions as to the particular height above O; in fact the initial height is undetermined. The slope will be the same, whatever the initial height. Let us therefore make a shot at what may be wanted, and start the sloping line at a height C above O. That is, we have the equation

$$y=ax+C.$$

It becomes evident now that in this case the added constant means the particular value that y has when $x=0$.

Now let us take a harder case, that of a line, the slope of which is not constant, but turns up more and more. Let us assume that the upward slope gets greater and greater in proportion as x grows. In symbols this is:

$$\frac{dy}{dx}=ax.$$

Or, to give a concrete case, take $a=\frac{1}{5}$, so that

$$\frac{dy}{dx}=\frac{1}{5}x.$$

Then we had best begin by calculating a few of the values of the slope at different values of x, and also draw little diagrams of them.

When $x=0,\quad \dfrac{dy}{dx}=0,$

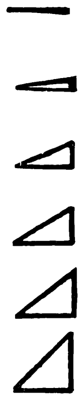
$x=1,\quad \dfrac{dy}{dx}=0{\cdot}2,$

$x=2,\quad \dfrac{dy}{dx}=0{\cdot}4,$

$x=3,\quad \dfrac{dy}{dx}=0{\cdot}6,$

$x=4,\quad \dfrac{dy}{dx}=0{\cdot}8,$

$x=5,\quad \dfrac{dy}{dx}=1{\cdot}0.$



Now try to put the pieces together, setting each so that the middle of its base is the proper distance to the right, and so that they fit together at the corners; thus (Fig. 49). The result is, of course, not a smooth

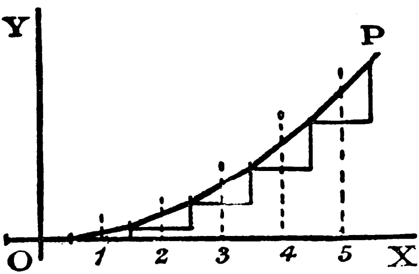


FIG. 49.

curve: but it is an approximation to one. If we had taken bits half as long, and twice as numerous, like Fig. 50, we should have a better approximation. But for a perfect curve we ought to take each dx and its corresponding dy infinitesimally small, and infinitely numerous.

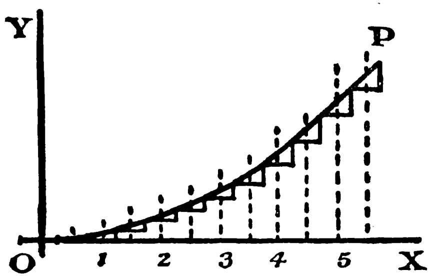


FIG. 50.

Then, how much ought the value of any y to be? Clearly, at any point P of the curve, the value of y will be the sum of all the little dy's from 0 up to that level, that is to say, $\int dy = y$. And as each dy is equal to $\frac{1}{5}x \cdot dx$, it follows that the whole y will be equal to the sum of all such bits as $\frac{1}{5}x \cdot dx$, or, as we should write it, $\int \frac{1}{5}x \cdot dx$.

Now if x had been constant, $\int \frac{1}{5}x \cdot dx$ would have been the same as $\frac{1}{5}x \int dx$, or $\frac{1}{5}x^2$. But x began by being 0, and increases to the particular value of x at the point P, so that its average value from 0 to that point is $\frac{1}{2}x$. Hence $\int \frac{1}{5}x dx = \frac{1}{10}x^2$; or $y = \frac{1}{10}x^2$.

But, as in the previous case, this requires the addition of an undetermined constant C, because we have not been told at what height above the origin the curve will begin, when $x=0$. So we write, as the equation of the curve drawn in Fig. 51,

$$y=\tfrac{1}{10}x^2+C.$$

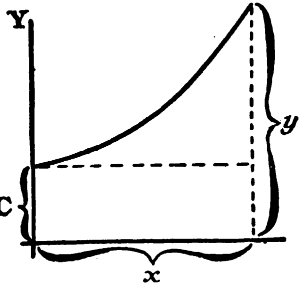


FIG. 51.

Exercises XVI. (See page 296 for Answers.)

(1) Find the ultimate sum of $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\text{etc.}$

(2) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}$ etc., is convergent, and find its sum to 8 terms.

(3) If $\log_\epsilon(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\text{etc.}$, find $\log_\epsilon 1{\cdot}3$.

(4) Following a reasoning similar to that explained in this chapter, find y,

$$(a)\ \text{if } \frac{dy}{dx}=\tfrac{1}{4}x;\quad (b)\ \text{if } \frac{dy}{dx}=\cos x.$$

(5) If $\dfrac{dy}{dx}=2x+3$, find y.

CHAPTER XVIII.

INTEGRATING AS THE REVERSE OF DIFFERENTIATING.

DIFFERENTIATING is the process by which when y is given us (as a function of x), we can find $\frac{dy}{dx}$.

Like every other mathematical operation, the process of differentiation may be reversed. Thus, if differentiating $y=x^4$ gives us $\frac{dy}{dx}=4x^3$, then, if one begins with $\frac{dy}{dx}=4x^3$, one would say that reversing the process would yield $y=x^4$. But here comes in a curious point. We should get $\frac{dy}{dx}=4x^3$ if we had begun with *any* of the following: x^4, or x^4+a, or x^4+c, or x^4 with *any* added constant. So it is clear that in working backwards from $\frac{dy}{dx}$ to y, one must make provision for the possibility of there being an added constant, the value of which will be undetermined until ascertained in some other way. So, if differentiating x^n yields nx^{n-1}, going backwards from $\frac{dy}{dx}=nx^{n-1}$ will give us $y=x^n+C$; where C stands for the yet undetermined possible constant.

Clearly, in dealing with powers of x, the rule for working backwards will be: Increase the power by 1, then divide by that increased power, and add the undetermined constant.

So, in the case where

$$\frac{dy}{dx}=x^n,$$

working backwards, we get

$$y=\frac{1}{n+1}x^{n+1}+C.$$

If differentiating the equation $y=ax^n$ gives us

$$\frac{dy}{dx}=anx^{n-1},$$

it is a matter of common sense that beginning with

$$\frac{dy}{dx}=anx^{n-1},$$

and reversing the process, will give us

$$y=ax^n.$$

So, when we are dealing with a multiplying constant, we must simply put the constant as a multiplier of the result of the integration.

Thus, if $\frac{dy}{dx}=4x^2$, the reverse process gives us $y=\frac{4}{3}x^3$.

But this is incomplete. For we must remember that if we had started with

$$y=ax^n+C,$$

where C is any constant quantity whatever, we should equally have found

$$\frac{dy}{dx}=anx^{n-1}.$$

So, therefore, when we reverse the process we must always remember to add on this undetermined constant, even if we do not yet know what its value will be.

This process, the reverse of differentiating, is called *integrating*; for it consists in finding the value of the whole quantity y when you are given only an expression for dy or for $\frac{dy}{dx}$. Hitherto we have as much as possible kept dy and dx together as a differential coefficient: henceforth we shall more often have to separate them.

If we begin with a simple case,

$$\frac{dy}{dx}=x^2.$$

We may write this, if we like, as

$$dy=x^2dx.$$

Now this is a "differential equation" which informs us that an element of y is equal to the corresponding element of x multiplied by x^2. Now, what we want is the integral; therefore, write down with the proper symbol the instructions to integrate both sides, thus:

$$\int dy = \int x^2 dx.$$

[Note as to reading integrals: the above would be read thus:

"*Integral dee-wy* equals *integral eks-squared dee-eks.*"]

We haven't yet integrated: we have only written down instructions to integrate—if we can. Let us try. Plenty of other fools can do it—why not we also? The left-hand side is simplicity itself. The sum of all the bits of y is the same thing as y itself. So we may at once put:

$$y = \int x^2 dx.$$

But when we come to the right-hand side of the equation we must remember that what we have got to sum up together is not all the dx's, but all such terms as $x^2 dx$; and this will *not* be the same as $x^2 \int dx$, because x^2 is not a constant. For some of the dx's will be multiplied by big values of x^2, and some will be multiplied by small values of x^2, according to what x happens to be. So we must bethink ourselves as to what we know about this process of integration being the reverse of differentiation. Now, our rule for this reversed process—see p. 191 *ante*—when dealing with x^n is "increase the power by one, and divide by the same number as this increased power."

That is to say, x^2dx will be changed* to $\frac{1}{3}x^3$. Put this into the equation; but don't forget to add the "constant of integration" C at the end. So we get:

$$u = \tfrac{1}{3}x^3 + C.$$

You have actually performed the integration. How easy!

Let us try another simple case.

Let $$\frac{dy}{dx} = ax^{12},$$

where a is any constant multiplier. Well, we found when differentiating (see p. 29) that any constant factor in the value of y reappeared unchanged in the value of $\frac{dy}{dx}$. In the reversed process of integrating, it will therefore also reappear in the value of y. So we may go to work as before, thus:

$$dy = ax^{12} \cdot dx,$$

$$\int dy = \int ax^{12} \cdot dx,$$

$$\int dy = a\int x^{12} dx,$$

$$y = a \times \tfrac{1}{13}x^{13} + C.$$

So that is done. How easy!

* You may ask: what has become of the little dx at the end? Well, remember that it was really part of the differential coefficient, and when changed over to the right-hand side, as in the x^2dx, serves as a reminder that x is the independent variable with respect to which the operation is to be effected; and, as the result of the product being totalled up, the power of x has increased by *one*. You will soon become familiar with all this.

We begin to realize now that integrating is a process *of finding our way back*, as compared with differentiating. If ever, during differentiating, we have found any particular expression—in this example ax^{12}—we can find our way back to the y from which it was derived. The contrast between the two processes may be illustrated by the following illustration due to a well-known teacher. If a stranger were set down in Trafalgar Square, and told to find his way to Euston Station, he might find the task hopeless. But if he had previously been personally conducted from Euston Station to Trafalgar Square, it would be comparatively easy to him to find his way back to Euston Station.

Integration of the Sum or Difference of two Functions.

Let $$\frac{dy}{dx}=x^2+x^3,$$

then $$dy=x^2dx+x^3dx.$$

There is no reason why we should not integrate each term separately: for, as may be seen on p. 35, we found that when we differentiated the sum of two separate functions, the differential coefficient was simply the sum of the two separate differentiations. So, when we work backwards, integrating, the integration will be simply the sum of the two separate integrations.

Our instructions will then be:

$$\int dy = \int (x^2 + x^3)\,dx$$
$$= \int x^2 dx + \int x^3 dx$$
$$y = \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + C.$$

If either of the terms had been a negative quantity, the corresponding term in the integral would have also been negative. So that differences are as readily dealt with as sums.

How to deal with Constant Terms.

Suppose there is in the expression to be integrated a constant term—such as this:

$$\frac{dy}{dx} = x^n + b.$$

This is laughably easy. For you have only to remember that when you differentiated the expression $y = ax$, the result was $\dfrac{dy}{dx} = a$. Hence, when you work the other way and integrate, the constant reappears multiplied by x. So we get

$$dy = x^n dx + b \cdot dx,$$
$$\int dy = \int x^n dx + \int b\,dx,$$
$$y = \frac{1}{n+1} x^{n+1} + bx + C.$$

Here are a lot of examples on which to try your newly acquired powers.

Examples.

(1) Given $\frac{dy}{dx}=24x^{11}$. Find y. *Ans.* $y=2x^{12}+C$.

(2) Find $\int(a+b)(x+1)dx$. It is $(a+b)\int(x+1)dx$

or $(a+b)\left[\int x\,dx+\int dx\right]$ or $(a+b)\left(\frac{x^2}{2}+x\right)+C$.

(3) Given $\frac{du}{dt}=gt^{\frac{1}{2}}$. Find u. *Ans.* $u=\frac{2}{3}gt^{\frac{3}{2}}+C$.

(4) $\frac{dy}{dx}=x^3-x^2+x$. Find y.

$dy=(x^3-x^2+x)dx$ or

$dy=x^3\,dx-x^2\,dx+x\,dx$; $y=\int x^3dx-\int x^2dx+\int x\,dx$;

and $$y=\tfrac{1}{4}x^4-\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+C.$$

(5) Integrate $9{\cdot}75x^{2{\cdot}25}dx$. *Ans.* $y=3x^{3{\cdot}25}+C$.

All these are easy enough. Let us try another case.

Let $$\frac{dy}{dx}=ax^{-1}.$$

Proceeding as before, we will write

$$dy=ax^{-1}\cdot dx,\quad \int dy=a\int x^{-1}dx.$$

Well, but what is the integral of $x^{-1}dx$?

If you look back amongst the results of differentiating x^2 and x^3 and x^n, etc., you will find we never got x^{-1} from any one of them as the value of $\frac{dy}{dx}$. We got $3x^2$ from x^3; we got $2x$ from x^2; we got 1 from x^1 (that is, from x itself); but we did not get x^{-1} from x^0, for two very good reasons. *First*, x^0 is simply $=1$, and is a constant, and could not have a differential coefficient. *Secondly*, even if it could be differentiated, its differential coefficient (got by slavishly following the usual rule) would be $0 \times x^{-1}$, and that multiplication by zero gives it zero value! Therefore when we now come to try to integrate $x^{-1}dx$, we see that it does not come in anywhere in the powers of x that are given by the rule:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}.$$

It is an exceptional case.

Well; but try again. Look through all the various differentials obtained from various functions of x, and try to find amongst them x^{-1}. A sufficient search will show that we actually did get $\frac{dy}{dx} = x^{-1}$ as the result of differentiating the function $y = \log_\epsilon x$ (see p. 148).

Then, of course, since we know that differentiating $\log_\epsilon x$ gives us x^{-1}, we know that, by reversing the process, integrating $dy = x^{-1}dx$ will give us $y = \log_\epsilon x$. But we must not forget the constant factor a that was given, nor must we omit to add the undetermined constant of integration. This then gives us as the solution to the present problem,

$$y = a \log_\epsilon x + C.$$

N.B.—Here note this very remarkable fact, that we could not have integrated in the above case if we had not happened to know the corresponding differentiation. If no one had found out that differentiating $\log_\epsilon x$ gave x^{-1}, we should have been utterly stuck by the problem how to integrate $x^{-1}dx$. Indeed it should be frankly admitted that this is one of the curious features of the integral calculus:—that you can't integrate anything before the reverse process of differentiating something else has yielded that expression which you want to integrate. No one, even to-day, is able to find the general integral of the expression,

$$\frac{dy}{dx} = a^{-x^2},$$

because a^{-x^2} has never yet been found to result from differentiating anything else.

Another simple case.

Find $\int (x+1)(x+2)dx$.

On looking at the function to be integrated, you remark that it is the product of two different functions of x. You could, you think, integrate $(x+1)dx$ by itself, or $(x+2)dx$ by itself. Of course you could. But what to do with a product? None of the differentiations you have learned have yielded you for the differential coefficient a product like this. Failing such, the simplest thing is to multiply up the two functions, and then integrate. This gives us

$$\int (x^2+3x+2)dx.$$

And this is the same as

$$\int x^2\,dx + \int 3x\,dx + \int 2dx.$$

And performing the integrations, we get

$$\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x + C.$$

Some other Integrals.

Now that we know that integration is the reverse of differentiation, we may at once look up the differential coefficients we already know, and see from what functions they were derived. This gives us the following integrals ready made:

$$x^{-1} \quad \text{(p. 148)}; \quad \int x^{-1}dx = \log_\epsilon x + C.$$

$$\frac{1}{x+a} \quad \text{(p. 149)}; \quad \int \frac{1}{x+a}dx = \log_\epsilon(x+a) + C.$$

$$\epsilon^x \quad \text{(p. 143)}; \quad \int \epsilon^x dx = \epsilon^x + C.$$

$$\epsilon^{-x} \quad \int \epsilon^{-x}dx = -\epsilon^{-x} + C$$

$$\left(\text{for if } y = -\frac{1}{\epsilon^x},\ \frac{dy}{dx} = -\frac{\epsilon^x \times 0 - 1 \times \epsilon^x}{\epsilon^{2x}} = \epsilon^{-x}\right).$$

$$\sin x \quad \text{(p. 168)}; \quad \int \sin x\, dx = -\cos x + C.$$

$$\cos x \quad \text{(p. 166)}; \quad \int \cos x\, dx = \sin x + C.$$

Also we may deduce the following:

$$\log_\epsilon x; \quad \int \log_\epsilon x\, dx = x(\log_\epsilon x - 1) + C$$

$$\left(\text{for if } y = x\log_\epsilon x - x,\ \frac{dy}{dx} = \frac{x}{x} + \log_\epsilon x - 1 = \log_\epsilon x\right).$$

$\log_{10} x$; $\quad \int \log_{10} x\, dx = 0{\cdot}4343x(\log_\epsilon x - 1) + C.$

a^x (p. 149); $\quad \int a^x dx = \dfrac{a^x}{\log_\epsilon a} + C.$

$\cos ax$; $\quad \int \cos ax\, dx = \dfrac{1}{a} \sin ax + C$

(for if $y = \sin ax$, $\dfrac{dy}{dx} = a \cos ax$; hence to get $\cos ax$ one must differentiate $y = \dfrac{1}{a} \sin ax$).

$\sin ax$; $\quad \int \sin ax\, dx = -\dfrac{1}{a} \cos ax + C.$

Try also $\cos^2\theta$; a little dodge will simplify matters:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1;$$

hence $$\cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1),$$

and $$\int \cos^2\theta\, d\theta = \tfrac{1}{2}\int (\cos 2\theta + 1)\, d\theta$$

$$= \tfrac{1}{2}\int \cos 2\theta\, d\theta + \tfrac{1}{2}\int d\theta.$$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$$ (See also p. 227.)

See also the Table of Standard Forms on pp. 286, 287. You should make such a table for yourself, putting in it only the general functions which you have successfully differentiated and integrated. See to it that it grows steadily!

On Double and Triple Integrals.

In many cases it is necessary to integrate some expression for two or more variables contained in it; and in that case the sign of integration appears more than once. Thus,

$$\iint f(x, y,)dx\,dy$$

means that some function of the variables x and y has to be integrated for each. It does not matter in which order they are done. Thus, take the function x^2+y^2. Integrating it with respect to x gives us:

$$\int (x^2+y^2)dx=\tfrac{1}{3}x^3+xy^2.$$

Now, integrate this with respect to y:

$$\int (\tfrac{1}{3}x^3+xy^2)dy=\tfrac{1}{3}x^3y+\tfrac{1}{3}xy^3,$$

to which of course a constant is to be added. If we had reversed the order of the operations, the result would have been the same.

In dealing with areas of surfaces and of solids, we have often to integrate both for length and breadth, and thus have integrals of the form

$$\iint u\cdot dx\,dy,$$

where u is some property that depends, at each point, on x and on y. This would then be called a *surface-integral.* It indicates that the value of all such elements as $u \cdot dx \cdot dy$ (that is to say, of the value of u over a little rectangle dx long and dy broad) has to be summed up over the whole length and whole breadth.

Similarly in the case of solids, where we deal with three dimensions. Consider any element of volume, the small cube whose dimensions are $dx\ dy\ dz$. If the figure of the solid be expressed by the function $f(x, y, z)$, then the whole solid will have the *volume-integral*,

$$\text{volume} = \iiint f(x, y, z) \cdot dx \cdot dy \cdot dz.$$

Naturally, such integrations have to be taken between appropriate limits* in each dimension; and the integration cannot be performed unless one knows in what way the boundaries of the surface depend on x, y, and z. If the limits for x are from x_1 to x_2, those for y from y_1 to y_2, and those for z from z_1 to z_2, then clearly we have

$$\text{volume} = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z) \cdot dx \cdot dy \cdot dz.$$

There are of course plenty of complicated and difficult cases; but, in general, it is quite easy to see the significance of the symbols where they are intended to indicate that a certain integration has to be performed over a given surface, or throughout a given solid space.

* See p. 208 for integration between limits.

Exercises XVII. (See p. 297 for the Answers.)

(1) Find $\int y\,dx$ when $y^2=4ax$.

(2) Find $\int \frac{3}{x^4}\,dx$.

(3) Find $\int \frac{1}{a} x^3 dx$.

(4) Find $\int (x^2+a)\,dx$.

(5) Integrate $5x^{-\frac{7}{2}}$.

(6) Find $\int (4x^3+3x^2+2x+1)\,dx$.

(7) If $\frac{dy}{dx}=\frac{ax}{2}+\frac{bx^2}{3}+\frac{cx^3}{4}$; find y.

(8) Find $\int \left(\frac{x^2+a}{x+a}\right) dx$.

(9) Find $\int (x+3)^3 dx$.

(10) Find $\int (x+2)(x-a)\,dx$.

(11) Find $\int (\sqrt{x}+\sqrt[3]{x})\,3a^2\,dx$.

(12) Find $\int (\sin\theta-\tfrac{1}{2})\frac{d\theta}{3}$.

(13) Find $\int \cos^2 a\theta\,d\theta$.

(14) Find $\int \sin^2\theta\,d\theta$.

(15) Find $\int \sin^2 a\theta\,d\theta$.

(16) Find $\int \epsilon^{3x} dx$.

(17) Find $\int \frac{dx}{1+x}$.

(18) Find $\int \frac{dx}{1-x}$.

CHAPTER XIX.

ON FINDING AREAS BY INTEGRATING.

ONE use of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let us try to get at the subject bit by bit.

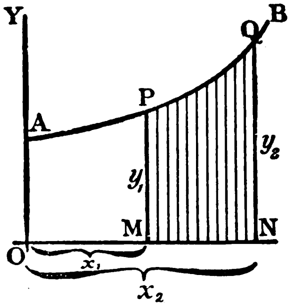


FIG. 52.

Let AB (Fig. 52) be a curve, the equation to which is known. That is, y in this curve is some known function of x. Think of a piece of the curve from the point P to the point Q.

Let a perpendicular PM be dropped from P, and another QN from the point Q. Then call $OM = x_1$ and $ON = x_2$, and the ordinates $PM = y_1$ and $QN = y_2$. We have thus marked out the area $PQNM$ that lies beneath the piece PQ. The problem is, *how can we calculate the value of this area?*

The secret of solving this problem is to conceive the area as being divided up into a lot of narrow strips, each of them being of the width dx. The smaller we take dx, the more of them there will be between x_1 and x_2. Now, the whole area is clearly equal to the sum of the areas of all such strips. Our business will then be to discover an expression for the area of any one narrow strip, and to integrate it so as to add together all the strips. Now think of any one of the strips. It will be like this: being bounded between two vertical sides, with a flat bottom dx, and with a slightly curved sloping top. Suppose we take its *average* height as being y; then, as its width is dx, its area will be $y\,dx$. And seeing that we may take the width as narrow as we please, if we

only take it narrow enough its average height will be the same as the height at the middle of it. Now let us call the unknown value of the whole area S, meaning surface The area of one strip will be simply a bit of the whole area, and may therefore be called dS. So we may write

$$\text{area of 1 strip} = dS = y \cdot dx.$$

If then we add up all the strips, we get

$$\text{total area } S = \int dS = \int y\,dx.$$

So then our finding S depends on whether we can integrate $y \cdot dx$ for the particular case, when we know what the value of y is as a function of x.

For instance, if you were told that for the particular curve in question $y = b + ax^2$, no doubt you could put that value into the expression and say: then I must find $\int (b + ax^2) dx$.

That is all very well; but a little thought will show you that something more must be done. Because the area we are trying to find is not the area under the whole length of the curve, but only the area limited on the left by PM, and on the right by QN, it follows that we must do something to define our area between those '*limits.*'

This introduces us to a new notion, namely that of *integrating between limits.* We suppose x to vary, and for the present purpose we do not require any value of x below x_1 (that is OM), nor any value of x above x_2 (that is ON). When an integral is to be thus defined between two limits, we call the lower of the two values *the inferior limit,* and the upper value *the superior limit.* Any integral so limited we designate as a *definite integral,* by way of distinguishing it from a *general integral* to which no limits are assigned.

In the symbols which give instructions to integrate, the limits are marked by putting them at the top and bottom respectively of the sign of integration. Thus the instruction

$$\int_{x=x_1}^{x=x_2} y \cdot dx$$

will be read: find the integral of $y \cdot dx$ between the inferior limit x_1 and the superior limit x_2.

Sometimes the thing is written more simply

$$\int_{x_1}^{x_2} y \cdot dx.$$

Well, but *how* do you find an integral between limits, when you have got these instructions?

Look again at Fig. 52 (p. 206). Suppose we could find the area under the larger piece of curve from A to Q, that is from $x=0$ to $x=x_2$, naming the area $AQNO$. Then, suppose we could find the area under the smaller piece from A to P, that is from $x=0$ to $x=x_1$, namely the area $APMO$. If then we were to subtract the smaller area from the larger, we should have left as a remainder the area $PQNM$, which is what we want. Here we have the clue as to what to do; the definite integral between the two limits is *the difference* between the integral worked out for the superior limit and the integral worked out for the lower limit.

Let us then go ahead. First, find the general integral thus:

$$\int y\, dx,$$

and, as $y=b+ax^2$ is the equation to the curve (Fig. 52),

$$\int (b+ax^2)\, dx$$

is the general integral which we must find.

Doing the integration in question by the rule (p. 196), we get

$$bx+\frac{a}{3}x^3+C;$$

and this will be the whole area from 0 up to any value of x that we may assign.

Therefore, the larger area up to the superior limit x_2 will be

$$bx_2+\frac{a}{3}x_2^3+C;$$

and the smaller area up to the inferior limit x_1 will be

$$bx_1+\frac{a}{3}x_1^3+C.$$

Now, subtract the smaller from the larger, and we get for the area S the value,

$$\text{area } S=b(x_2-x_1)+\frac{a}{3}(x_2^3-x_1^3).$$

This is the answer we wanted. Let us give some numerical values. Suppose $b=10$, $a=0{\cdot}06$, and $x_2=8$ and $x_1=6$. Then the area S is equal to

$$\begin{aligned}
&10(8-6)+\frac{0{\cdot}06}{3}(8^3-6^3)\\
&=20+0{\cdot}02(512-216)\\
&=20+0{\cdot}02\times 296\\
&=20+5{\cdot}92\\
&=25{\cdot}92.
\end{aligned}$$

Let us here put down a symbolic way of stating what we have ascertained about limits:

$$\int_{x=x_1}^{x=x_2} y\,dx=y_2-y_1,$$

where y_2 is the integrated value of $y\,dx$ corresponding to x_2, and y_1 that corresponding to x_1.

All integration between limits requires the difference between two values to be thus found. Also note that, in making the subtraction the added constant C has disappeared.

Examples.

(1) To familiarize ourselves with the process, let us take a case of which we know the answer beforehand. Let us find the area of the triangle (Fig. 53), which

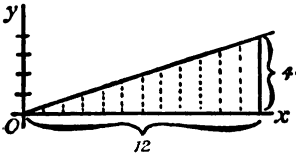


FIG. 53.

has base $x=12$ and height $y=4$. We know beforehand, from obvious mensuration, that the answer will come 24.

Now, here we have as the "curve" a sloping line for which the equation is

$$y=\frac{x}{3}.$$

The area in question will be

$$\int_{x=0}^{x=12} y \cdot dx = \int_{x=0}^{x=12} \frac{x}{3} \cdot dx.$$

Integrating $\frac{x}{3}dx$ (p. 194), and putting down the value of the general integral in square brackets with the limits marked above and below, we get

$$\text{area} = \left[\frac{1}{3}\cdot\frac{1}{2}x^2 + C\right]_{x=0}^{x=12}$$

$$= \left[\frac{x^2}{6} + C\right]_{x=0}^{x=12}$$

$$= \left[\frac{12^2}{6} + C\right] - \left[\frac{0^2}{6} + C\right]$$

$$= \frac{144}{6} = 24. \quad \textit{Ans.}$$

Note that, in dealing with definite integrals, the constant C always disappears by subtraction.

Let us satisfy ourselves about this rather surprising dodge of calculation, by testing it on a simple example. Get some squared paper, preferably some that is ruled in little squares of one-eighth inch or one-tenth inch each way. On this squared paper plot out the graph of this equation,

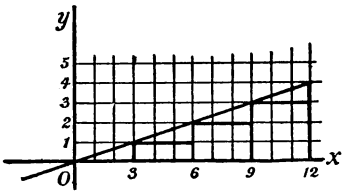


FIG. 54.

$$y = \frac{x}{3}.$$

The values to be plotted will be:

x	0	3	6	9	12
y	0	1	2	3	4

The plot is given in Fig. 54.

Now reckon out the area beneath the curve *by counting the little squares* below the line, from $x=0$ as far as $x=12$ on the right. There are 18 whole squares and four triangles, each of which has an area equal to $1\frac{1}{2}$ squares; or, in total, 24 squares. Hence 24 is the numerical value of the integral of $\frac{x}{3}dx$ between the lower limit of $x=0$ and the higher limit of $x=12$.

As a further exercise, show that the value of the same integral between the limits of $x=3$ and $x=15$ is 36.

(2) Find the area, between limits $x=x_1$ and $x=0$, of the curve $y=\dfrac{b}{x+a}$.

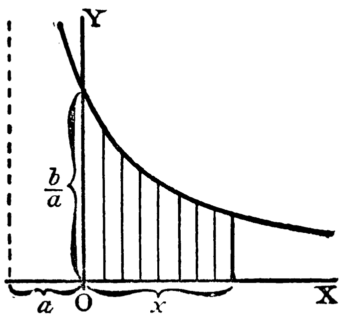


FIG. 55.

$$\text{Area}=\int_{x=0}^{x=x_1} y \cdot dx=\int_{x=0}^{x=x_1} \frac{b}{x+a}\, dx$$

$$=b\Big[\log_\epsilon(x+a)+C\Big]_0^{x_1}$$

$$=b[\log_\epsilon(x_1+a)+C-\log_\epsilon(0+a)-C]$$

$$=b\log_\epsilon\frac{x_1+a}{a}.\quad \textit{Ans.}$$

Let it be noted that this process of subtracting one part from a larger to find the difference is really a common practice. How do you find the area of a

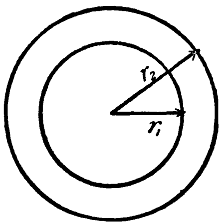


FIG. 56.

plane ring (Fig. 56), the outer radius of which is r_2 and the inner radius is r_1? You know from mensuration that the area of the outer circle is πr_2^2; then you find the area of the inner circle, πr_1^2; then you subtract the latter from the former, and find area of ring $=\pi(r_2^2-r_1^2)$; which may be written

$$\pi(r_2+r_1)(r_2-r_1)$$

= mean circumference of ring × width of ring.

(3) Here's another case—that of *the die-away curve*

(p. 156). Find the area between $x=0$ and $x=a$, of the curve (Fig. 57) whose equation is

$$y=b\epsilon^{-x}.$$

$$\text{Area}=b\int_{x=0}^{x=a}\epsilon^{-x}\cdot dx.$$

The integration (p. 201) gives

$$=b\Big[-\epsilon^{-x}\Big]_0^a$$

$$=b[-\epsilon^{-a}-(-\epsilon^{-0})]$$

$$=b(1-\epsilon^{-a}).$$

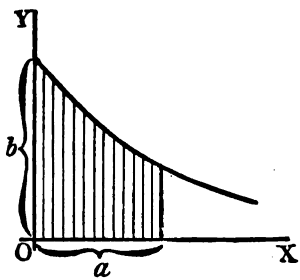


FIG. 57.

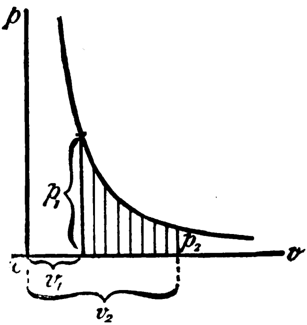


FIG. 58.

(4) Another example is afforded by the adiabatic curve of a perfect gas, the equation to which is $pv^n=c$, where p stands for pressure, v for volume, and n is of the value 1·42 (Fig. 58).

Find the area under the curve (which is proportional to the work done in suddenly compressing the gas) from volume v_2 to volume v_1.

Here we have

$$\begin{aligned}
\text{area} &= \int_{v=v_1}^{v=v_2} cv^{-n} \cdot dv \\
&= c\left[\frac{1}{1-n} v^{1-n}\right]_{v_1}^{v_2} \\
&= c\frac{1}{1-n}(v_2^{1-n} - v_1^{1-n}) \\
&= \frac{-c}{0{\cdot}42}\left(\frac{1}{v_2^{0{\cdot}42}} - \frac{1}{v_1^{0{\cdot}42}}\right).
\end{aligned}$$

An Exercise.

Prove the ordinary mensuration formula, that the area A of a circle whose radius is R, is equal to πR^2.

Consider an elementary zone or annulus of the surface (Fig. 59), of breadth dr, situated at a distance

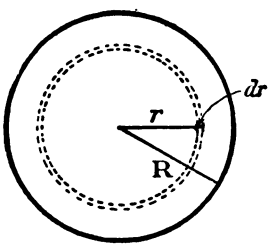


FIG. 59.

r from the centre. We may consider the entire surface as consisting of such narrow zones, and the whole area A will simply be the integral of all such elementary zones from centre to margin, that is, integrated from $r=0$ to $r=R$.

We have therefore to find an expression for the elementary area dA of the narrow zone. Think of it as a strip of breadth dr, and of a length that is the periphery of the circle of radius r, that is, a length of $2\pi r$. Then we have, as the area of the narrow zone,

$$dA = 2\pi r\,dr.$$

Hence the area of the whole circle will be:

$$A = \int dA = \int_{r=0}^{r=R} 2\pi r \cdot dr = 2\pi \int_{r=0}^{r=R} r \cdot dr.$$

Now, the general integral of $r \cdot dr$ is $\frac{1}{2}r^2$. Therefore,

$$A = 2\pi\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R};$$

or

$$A = 2\pi\left[\tfrac{1}{2}R^2 - \tfrac{1}{2}(0)^2\right];$$

whence

$$A = \pi R^2.$$

Another Exercise.

Let us find the mean ordinate of the positive part of the curve $y = x - x^2$, which is shown in Fig. 60.

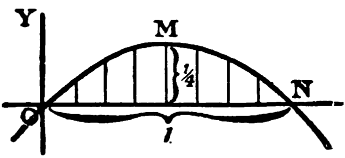


FIG. 60.

To find the mean ordinate, we shall have to find the area of the piece OMN, and then divide it by the length of the base ON. But before we can find the area we must ascertain the length of the base, so as to know up to what limit we are to integrate.

At N the ordinate y has zero value; therefore, we must look at the equation and see what value of x will make $y=0$. Now, clearly, if x is 0, y will also be 0, the curve passing through the origin O; but also, if $x=1$, $y=0$: so that $x=1$ gives us the position of the point N.

Then the area wanted is

$$=\int_{x=0}^{x=1}(x-x^2)\,dx=\left[\tfrac{1}{2}x^3-\tfrac{1}{3}x^3\right]_0^1=[\tfrac{1}{2}-\tfrac{1}{3}]-[0-0]=\tfrac{1}{6}.$$

But the base length is 1.

Therefore, the average ordinate of the curve $=\frac{1}{6}$.

[*N.B.*—It will be a pretty and simple exercise in maxima and minima to find by differentiation what is the height of the maximum ordinate. It *must* be greater than the average.]

The mean ordinate of any curve, over a range from $x=0$ to $x=x_1$, is given by the expression,

$$\text{mean } y=\frac{1}{x_1}\int_{x=0}^{x=x_1} y\cdot dx.$$

If the mean ordinate be required over a distance not beginning at the origin but beginning at a point distant x_1 from the origin and ending at a point distant x_2 from the origin, the value will be

$$\text{mean } y=\frac{1}{x_2-x_1}\int_{x=x_1}^{x=x_2} y\,dx.$$

Areas in Polar Coordinates.

When the equation of the boundary of an area is given as a function of the distance r of a point of it from a fixed point O (see Fig. 61) called the *pole*, and

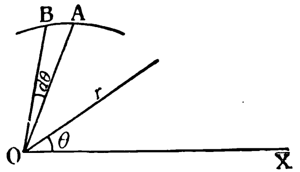


FIG. 61.

of the angle which r makes with the positive horizontal direction OX, the process just explained can be applied just as easily, with a small modification. Instead of a strip of area, we consider a small triangle OAB, the angle at O being $d\theta$, and we find the sum of all the little triangles making up the required area.

The area of such a small triangle is approximately $\frac{AB}{2}\times r$ or $\frac{r\,d\theta}{2}\times r$; hence the portion of the area included between the curve and two positions of r corresponding to the angles θ_1 and θ_2 is given by

$$\frac{1}{2}\int_{\theta=\theta_1}^{\theta=\theta_2} r^2\,d\theta.$$

Examples.

(1) Find the area of the sector of 1 radian in a circumference of radius a inch.

The polar equation of the circumference is evidently $r=a$. The area is

$$\tfrac{1}{2}\int_{\theta=0}^{\theta=1} a^2\,d\theta=\frac{a^2}{2}\int_{\theta=0}^{\theta=1} d\theta=\frac{a^2}{2}.$$

(2) Find the area of the first quadrant of the curve (known as "Pascal's Snail"), the polar equation of which is $r=a(1+\cos\theta)$.

$$\text{Area}=\tfrac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} a^2(1+\cos\theta)^2\,d\theta$$

$$=\frac{a^2}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=\frac{a^2}{2}\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{a^2(3\pi+8)}{8}.$$

Volumes by Integration.

What we have done with the area of a little strip of a surface, we can, of course, just as easily do with the volume of a little strip of a solid. We can add up all the little strips that make up the total solid, and find its volume, just as we have added up all the small little bits that made up an area to find the final area of the figure operated upon.

Examples.

(1) Find the volume of a sphere of radius r.

A thin spherical shell has for volume $4\pi x^2 dx$ (see Fig. 59, p. 216); summing up all the concentric shells which make up the sphere, we have

$$\text{volume sphere} = \int_{x=0}^{x=r} 4\pi x^2 dx = 4\pi \left[\frac{x^3}{3}\right]_0^r = \tfrac{4}{3}\pi r^3.$$

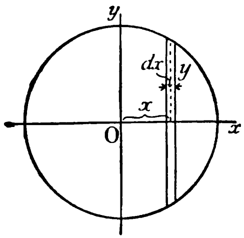


FIG. 62.

We can also proceed as follows: a slice of the sphere, of thickness dx, has for volume $\pi y^2 dx$ (see Fig. 62). Also x and y are related by the expression

$$y^2 = r^2 - x^2.$$

Hence

$$\text{volume sphere} = 2\int_{x=0}^{x=r} \pi(r^2 - x^2)dx$$

$$= 2\pi\left[\int_{x=0}^{x=r} r^2 dx - \int_{x=0}^{x=r} x^2 dx\right]$$

$$= 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{4\pi}{3} r^3.$$

(2) Find the volume of the solid generated by the revolution of the curve $y^2 = 6x$ about the axis of x, between $x = 0$ and $x = 4$.

The volume of a slice of the solid is $\pi y^2 dx$.

Hence volume $= \int_{x=0}^{x=4} \pi y^2\, dx = 6\pi \int_{x=0}^{x=4} x\, dx$

$$= 6\pi \left[\frac{x^2}{2} \right]_0^4 = 48\pi = 150{\cdot}8.$$

On Quadratic Means.

In certain branches of physics, particularly in the study of alternating electric currents, it is necessary to be able to calculate the *quadratic mean* of a variable quantity. By "quadratic mean" is denoted the square root of the mean of the squares of all the values between the limits considered. Other names for the quadratic mean of any quantity are its "virtual" value, or its "R.M.S." (meaning root-mean-square) value. The French term is *valeur efficace.* If y is the function under consideration, and the quadratic mean is to be taken between the limits of $x=0$ and $x=l$; then the quadratic mean is expressed as

$$\sqrt[2]{\frac{1}{l}\int_0^l y^2\, dx}.$$

Examples.

(1) To find the quadratic mean of the function $y=ax$ (Fig. 63).

Here the integral is $\int_0^l a^2x^2\, dx$,

which is $\frac{1}{3}a^2l^3$.

Dividing by l and taking the square root, we have

$$\text{quadratic mean} = \frac{1}{\sqrt{3}}\, al.$$

Here the arithmetical mean is $\frac{1}{2}al$; and the ratio of quadratic to arithmetical mean (this ratio is called the *form-factor*) is $\frac{2}{\sqrt{3}} = 1{\cdot}155$.

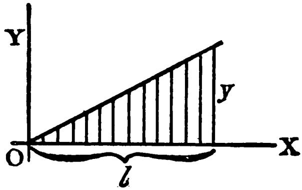


FIG. 63.

(2) To find the quadratic mean of the function $y = x^a$.

The integral is $\int_{x=0}^{x=l} x^{2a}\,dx$, that is $\frac{l^{2a+1}}{2a+1}$.

Hence quadratic mean $= \sqrt[2]{\frac{l^{2a}}{2a+1}}$.

(3) To find the quadratic mean of the function $y = a^{\frac{x}{2}}$.

The integral is $\int_{x=0}^{x=l} \left(a^{\frac{x}{2}}\right)^2 dx$, that is $\int_{x=0}^{x=l} a^x dx$,

or $$\left[\frac{a^x}{\log_\epsilon a}\right]_{x=0}^{x=l};$$

which is $\frac{a^l - 1}{\log_\epsilon a}$.

Hence the quadratic mean is $\sqrt[2]{\frac{a^l - 1}{l \log_\epsilon a}}$.

Exercises XVIII. (See p. 297 for Answers.)

(1) Find the area of the curve $y=x^2+x-5$ between $x=0$ and $x=6$, and the mean ordinate between these limits.

(2) Find the area of the parabola $y=2a\sqrt{x}$ between $x=0$ and $x=a$. Show that it is two-thirds of the rectangle of the limiting ordinate and of its abscissa.

(3) Find the area of the positive portion of a sine curve and the mean ordinate.

(4) Find the area of the portion of the curve $y=\sin^2 x$ from $0°$ to $180°$, and find the mean ordinate.

(5) Find the area included between the two branches of the curve $y=x^2 \pm x^{\frac{5}{2}}$ from $x=0$ to $x=1$, also the area of the positive portion of the lower branch of the curve (see Fig. 30, p. 108).

(6) Find the volume of a cone of radius of base r, and of height h.

(7) Find the area of the curve $y=x^3-\log_\epsilon x$ between $x=0$ and $x=1$.

(8) Find the volume generated by the curve $y=\sqrt{1+x^2}$, as it revolves about the axis of x, between $x=0$ and $x=4$.

(9) Find the volume generated by a sine curve revolving about the axis of x.

(10) Find the area of the portion of the curve $xy=a$ included between $x=1$ and $x=a$. Find the mean ordinate between these limits.

(11) Show that the quadratic mean of the function $y=\sin x$, between the limits of 0 and π radians, is $\frac{\sqrt{2}}{2}$. Find also the arithmetical mean of the same function between the same limits; and show that the form-factor is $=1{\cdot}11$.

(12) Find the arithmetical and quadratic means of the function x^2+3x+2, from $x=0$ to $x=3$.

(13) Find the quadratic mean and the arithmetical mean of the function $y=A_1 \sin x+A_3 \sin 3x$.

(14) A certain curve has the equation $y=3{\cdot}42\epsilon^{0{\cdot}21x}$. Find the area included between the curve and the axis of x, from the ordinate at $x=2$ to the ordinate at $x=8$. Find also the height of the mean ordinate of the curve between these points.

(15) Show that the radius of a circle, the area of which is twice the area of a polar diagram, is equal to the quadratic mean of all the values of r for that polar diagram.

(16) Find the volume generated by the curve $y=\pm\frac{x}{6}\sqrt{x(10-x)}$ rotating about the axis of x.

CHAPTER XX.

DODGES, PITFALLS, AND TRIUMPHS.

Dodges. A great part of the labour of integrating things consists in licking them into some shape that can be integrated. The books—and by this is meant the serious books—on the Integral Calculus are full of plans and methods and dodges and artifices for this kind of work. The following are a few of them.

Integration by Parts. This name is given to a dodge, the formula for which is

$$\int u\,dx = ux - \int x\,du + C.$$

It is useful in some cases that you can't tackle directly, for it shows that if in any case $\int x\,du$ can be found, then $\int u\,dx$ can also be found. The formula can be deduced as follows. From p. 38, we have,

$$d(ux) = u\,dx + x\,du,$$

which may be written

$$u\,dx = d(ux) - x\,du,$$

which by direct integration gives the above expression.

Examples.

(1) Find $\int w \cdot \sin w\, dw$.

Write $u = w$, and for $\sin w \cdot dw$ write dx. We shall then have $du = dw$, while $\int \sin w \cdot dw = -\cos w = x$.

Putting these into the formula, we get

$$\int w \cdot \sin w\, dw = w(-\cos w) - \int -\cos w\, dw$$
$$= -w \cos w + \sin w + C.$$

(2) Find $\int x\epsilon^x\, dx$.

Write $\quad u = x, \quad \epsilon^x\, dx = dv,$

then $\quad du = dx, \quad v = \epsilon^x,$

and $\quad \int x\epsilon^x\, dx = x\epsilon^x - \int \epsilon^x\, dx$ (by the formula)

$$= x\epsilon^x - \epsilon^x = \epsilon^x(x-1) + C.$$

(3) Try $\int \cos^2\theta\, d\theta$.

$$u = \cos\theta, \quad \cos\theta\, d\theta = dv.$$

Hence $\quad du = -\sin\theta\, d\theta, \quad v = \sin\theta,$

$$\int \cos^2\theta\, d\theta = \cos\theta \sin\theta + \int \sin^2\theta\, d\theta$$
$$= \frac{2\cos\theta\sin\theta}{2} + \int (1-\cos^2\theta)\, d\theta$$
$$= \frac{\sin 2\theta}{2} + \int d\theta - \int \cos^2\theta\, d\theta.$$

Hence $\quad 2\int \cos^2\theta\, d\theta = \frac{\sin 2\theta}{2} + \theta$

and $\quad \int \cos^2\theta\, d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$

(4) Find $\int x^2 \sin x\,dx$.

Write $\qquad x^2 = u, \quad \sin x\,dx = dv;$
then $\qquad du = 2x\,dx, \qquad v = -\cos x,$

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx.$$

Now find $\int x \cos x\,dx$, integrating by parts (as in Example 1 above):

$$\int x \cos x\,dx = x \sin x + \cos x + C.$$

Hence

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C'$$

$$= 2\left[x \sin x + \cos x\left(1 - \frac{x^2}{2}\right)\right] + C'.$$

(5) Find $\int \sqrt{1-x^2}\,dx$.

Write $\qquad u = \sqrt{1-x^2}, \quad dx = dv;$

then $\qquad du = -\dfrac{x\,dx}{\sqrt{1-x^2}}$ (see Chap. IX., p. 67)

and $x = v$; so that

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

Here we may use a little dodge, for we can write

$$\int \sqrt{1-x^2}\,dx = \int \frac{(1-x^2)\,dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

Adding these two last equations, we get rid of $\int \dfrac{x^2\,dx}{\sqrt{1-x^2}}$, and we have

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{dx}{\sqrt{1-x^2}}.$$

Do you remember meeting $\dfrac{dx}{\sqrt{1-x^2}}$? it is got by differentiating $y=\arc\sin x$ (see p. 171); hence its integral is $\arc\sin x$, and so

$$\int\sqrt{1-x^2}\,dx=\frac{x\sqrt{1-x^2}}{2}+\tfrac{1}{2}\arc\sin x+C.$$

You can try now some exercises by yourself; you will find some at the end of this chapter.

Substitution. This is the same dodge as explained in Chap. IX., p. 67. Let us illustrate its application to integration by a few examples.

(1) $\int\sqrt{3+x}\,dx.$

Let $\qquad 3+x=u, \quad dx=du;$

replace $\qquad \int u^{\frac{1}{2}}du=\frac{2}{3}u^{\frac{3}{2}}=\frac{2}{3}(3+x)^{\frac{3}{2}}.$

(2) $\int\dfrac{dx}{\epsilon^x+\epsilon^{-x}}.$

Let $\qquad \epsilon^x=u, \quad \dfrac{du}{dx}=\epsilon^x, \text{ and } dx=\dfrac{du}{\epsilon^x};$

so that $\int\dfrac{dx}{\epsilon^x+\epsilon^{-x}}=\int\dfrac{du}{\epsilon^x(\epsilon^x+\epsilon^{-x})}=\int\dfrac{du}{u\left(u+\dfrac{1}{u}\right)}=\int\dfrac{du}{u^2+1}.$

$\dfrac{du}{1+u^2}$ is the result of differentiating $\arc\tan u$.

Hence the integral is $\arc\tan\epsilon^x$.

(3) $\int\dfrac{dx}{x^2+2x+3}=\int\dfrac{dx}{x^2+2x+1+2}=\int\dfrac{dx}{(x+1)^2+(\sqrt{2})^2}.$

Let $\quad x+1=u, \quad dx=du;$

then the integral becomes $\int\frac{du}{u^2+(\sqrt{2})^2}$; but $\frac{du}{u^2+a^2}$ is the result of differentiating $\frac{1}{a}\arctan\frac{u}{a}$.

Hence one has finally $\frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}$ for the value of the given integral.

Formulæ of Reduction are special forms applicable chiefly to binomial and trigonometrical expressions that have to be integrated, and have to be reduced into some form of which the integral is known.

Rationalization, and *Factorization of Denominator* are dodges applicable in special cases, but they do not admit of any short or general explanation. Much practice is needed to become familiar with these preparatory processes.

The following example shows how the process of splitting into partial fractions, which we learned in Chap. XIII., p. 122, can be made use of in integration.

Take again $\int\frac{dx}{x^2+2x+3}$; if we split $\frac{1}{x^2+2x+3}$ into partial fractions, this becomes (see p. 232):

$$\frac{1}{2\sqrt{-2}}\left[\int\frac{dx}{x+1-\sqrt{-2}}-\int\frac{dx}{x+1+\sqrt{-2}}\right]$$

$$=\frac{1}{2\sqrt{-2}}\log_\epsilon\frac{x+1-\sqrt{-2}}{x+1+\sqrt{-2}}.$$

Notice that the same integral can be expressed sometimes in more than one way (which are equivalent to one another).

Pitfalls. A beginner is liable to overlook certain points that a practised hand would avoid; such as the use of factors that are equivalent to either zero or infinity, and the occurrence of indeterminate quantities such as $\frac{0}{0}$. There is no golden rule that will meet every possible case. Nothing but practice and intelligent care will avail. An example of a pitfall which had to be circumvented arose in Chap. XVIII., p. 199, when we came to the problem of integrating $x^{-1}\,dx$.

Triumphs. By triumphs must be understood the successes with which the calculus has been applied to the solution of problems otherwise intractable. Often in the consideration of physical relations one is able to build up an expression for the law governing the interaction of the parts or of the forces that govern them, such expression being naturally in the form of a *differential equation,* that is an equation containing differential coefficients with or without other algebraic quantities. And when such a differential equation has been found, one can get no further until it has been integrated. Generally it is much easier to state the appropriate differential equation than to solve it: the real trouble begins then only when one wants to integrate, unless indeed the equation is seen to possess some standard form of which the integral is known, and then the triumph is easy. The equation which results from integrating a differential equation is called* its "solution"; and it is quite astonishing how in many cases the solution looks as if it had no relation to the differential equation of which it is the integrated form. The solution often seems as different from the original expression as a butterfly does from the caterpillar that it was. Who would have supposed that such an innocent thing as

$$\frac{dy}{dx} = \frac{1}{a^2 - x^2}$$

could blossom out into

$$y = \frac{1}{2a} \log_\epsilon \frac{a+x}{a-x} + C?$$

yet the latter is the *solution* of the former.

As a last example, let us work out the above together By partial fractions,

$$\frac{1}{a^2-x^2} = \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)},$$

$$dy = \frac{dx}{2a(a+x)} + \frac{dx}{2a(a-x)},$$

$$y = \frac{1}{2a}\left(\int \frac{dx}{a+x} + \int \frac{dx}{a-x}\right)$$

$$= \frac{1}{2a}(\log_\epsilon(a+x) - \log_\epsilon(a-x))$$

$$= \frac{1}{2a}\log_\epsilon \frac{a+x}{a-x} + C.$$

*This means that the actual result of solving it is called its "solution." But many mathematicians would say, with Professor Forsyth, "every differential equation *is considered as solved* when the value of the dependent variable is expressed as a function of the independent variable by means either of known functions, or of integrals, whether the integrations in the latter can or cannot be expressed in terms of functions already known."

Not a very difficult metamorphosis!

There are whole treatises, such as Boole's *Differential Equations*, devoted to the subject of thus finding the "solutions" for different original forms.

Exercises XIX. (See p. 298 for Answers.)

(1) Find $\int \sqrt{a^2-x^2}\,dx$. (2) Find $\int x \log_\epsilon x\,dx$.

(3) Find $\int x^a \log_\epsilon x\,dx$. (4) Find $\int \epsilon^x \cos \epsilon^x\,dx$.

(5) Find $\int \frac{1}{x} \cos(\log_\epsilon x)\,dx$. (6) Find $\int x^2 \epsilon^x\,dx$.

(7) Find $\int \frac{(\log_\epsilon x)^a}{x}\,dx$. (8) Find $\int \frac{dx}{x \log_\epsilon x}$.

(9) Find $\int \frac{5x+1}{x^2+x-2}\,dx$. (10) Find $\int \frac{(x^2-3)\,dx}{x^3-7x+6}$.

(11) Find $\int \frac{b\,dx}{x^2-a^2}$. (12) Find $\int \frac{4x\,dx}{x^4-1}$.

(13) Find $\int \frac{dx}{1-x^4}$. (14) Find $\int \frac{dx}{x\sqrt{a-bx^2}}$.

CHAPTER XXI.

FINDING SOLUTIONS.

In this chapter we go to work finding solutions to some important differential equations, using for this purpose the processes shown in the preceding chapters.

The beginner, who now knows how easy most of those processes are in themselves, will here begin to realize that integration is *an art.* As in all arts, so in this, facility can be acquired only by diligent and regular practice. He who would attain that facility must work out examples, and more examples, and yet more examples, such as are found abundantly in all the regular treatises on the Calculus. Our purpose here must be to afford the briefest introduction to serious work.

Example 1. Find the solution of the differential equation
$$ay + b\frac{dy}{dx} = 0.$$

Transposing we have
$$b\frac{dy}{dx} = -ay.$$

Now the mere inspection of this relation tells us that we have got to do with a case in which $\frac{dy}{dx}$ is proportional to y. If we think of the curve which will represent y as a function of x, it will be such that its slope at any point will be proportional to the ordinate at that point, and will be a negative slope if y is positive. So obviously the curve will be a die-away curve (p. 156), and the solution will contain ϵ^{-x} as a factor. But, without presuming on this bit of sagacity, let us go to work.

As both y and dy occur in the equation and on opposite sides, we can do nothing until we get both y and dy to one side, and dx to the other. To do this, we must split our usually inseparable companions dy and dx from one another.

$$\frac{dy}{y} = -\frac{a}{b}dx.$$

Having done the deed, we now can see that both sides have got into a shape that is integrable, because we recognize $\frac{dy}{y}$, or $\frac{1}{y}dy$, as a differential that we have met with (p. 147) when differentiating logarithms. So we may at once write down the instructions to integrate,

$$\int \frac{dy}{y} = \int -\frac{a}{b}dx;$$

and doing the two integrations, we have:

$$\log_\epsilon y = -\frac{a}{b}x + \log_\epsilon C,$$

where $\log_\epsilon C$ is the yet undetermined constant* of integration. Then, delogarizing, we get:

$$y = C\epsilon^{-\frac{a}{b}x},$$

which is *the solution* required. Now, this solution looks quite unlike the original differential equation from which it was constructed: yet to an expert mathematician they both convey the same information as to the way in which y depends on x.

Now, as to the C, its meaning depends on the initial value of y. For if we put $x=0$ in order to see what value y then has, we find that this makes $y=C\epsilon^{-0}$; and as $\epsilon^{-0}=1$, we see that C is nothing else than the particular value† of y at starting. This we may call y_0, and so write the solution as

$$y = y_0\epsilon^{-\frac{a}{b}x}.$$

Example 2.

Let us take as an example to solve

$$ay + b\frac{dy}{dx} = g,$$

where g is a constant. Again, inspecting the equation will suggest, (1) that somehow or other ϵ^x will come into the solution, and (2) that if at any part of the

* We may write down any form of constant as the "constant of integration," and the form $\log_\epsilon C$ is adopted here by preference, because the other terms in this line of equation are, or are treated as logarithms; and it saves complications afterward if the added constant be *of the same kind.*

† Compare what was said about the "constant of integration," with reference to Fig. 48 on p. 187, and Fig. 51 on p. 190.

curve y becomes either a maximum or a minimum, so that $\frac{dy}{dx}=0$, then y will have the value $=\frac{g}{a}$. But let us go to work as before, separating the differentials and trying to transform the thing into some integrable shape.

$$b\frac{dy}{dx}=g-ay;$$

$$\frac{dy}{dx}=\frac{a}{b}\left(\frac{g}{a}-y\right);$$

$$\frac{dy}{y-\frac{g}{a}}=-\frac{a}{b}dx.$$

Now we have done our best to get nothing but y and dy on one side, and nothing but dx on the other. But is the result on the left side integrable?

It is of the same form as the result on p. 148; so, writing the instructions to integrate, we have:

$$\int\frac{dy}{y-\frac{g}{a}}=-\int\frac{a}{b}dx;$$

and, doing the integration, and adding the appropriate constant,

$$\log_\epsilon\left(y-\frac{g}{a}\right)=-\frac{a}{b}x+\log_\epsilon C;$$

whence $$y-\frac{g}{a}=C\epsilon^{-\frac{a}{b}x};$$

and finally, $$y=\frac{g}{a}+C\epsilon^{-\frac{a}{b}x}.$$

which is *the solution.*

If the condition is laid down that $y=0$ when $x=0$ we can find C; for then the exponential becomes $=1$; and we have

$$0=\frac{g}{a}+C,$$

or

$$C=-\frac{g}{a}.$$

Putting in this value, the solution becomes

$$y=\frac{g}{a}(1-\epsilon^{-\frac{a}{b}x}).$$

But further, if x grows indefinitely, y will grow to a maximum; for when $x=\infty$, the exponential $=0$, giving $y_{\text{max.}}=\frac{g}{a}$. Substituting this, we get finally

$$y=y_{\text{max.}}(1-\epsilon^{-\frac{a}{b}x}).$$

This result is also of importance in physical science.

Example 3.

Let

$$ay+b\frac{dy}{dt}=g\cdot\sin 2\pi nt.$$

We shall find this much less tractable than the preceding. First divide through by b.

$$\frac{dy}{dt}+\frac{a}{b}y=\frac{g}{b}\sin 2\pi nt.$$

Now, as it stands, the left side is not integrable. But it can be made so by the artifice—and this is where skill and practice suggest a plan—of multiplying all the terms by $\epsilon^{\frac{a}{b}t}$, giving us:

$$\frac{dy}{dt}\epsilon^{\frac{a}{b}t}+\frac{a}{b}y\epsilon^{\frac{a}{b}t}=\frac{g}{b}\epsilon^{\frac{a}{b}t}\cdot\sin 2\pi nt,$$

which is the same as

$$\frac{dy}{dt}\epsilon^{\frac{a}{b}t}+y\frac{d(\epsilon^{\frac{a}{b}t})}{dt}=\frac{g}{b}\epsilon^{\frac{a}{b}t}\cdot\sin 2\pi nt;$$

and this being a perfect differential may be integrated thus:—since, if $u=y\epsilon^{\frac{a}{b}t}$, $\frac{du}{dt}=\frac{dy}{dt}\epsilon^{\frac{a}{b}t}+y\frac{d\left(\epsilon^{\frac{a}{b}t}\right)}{dt}$,

$$y\epsilon^{\frac{a}{b}t}=\frac{g}{b}\int\epsilon^{\frac{a}{b}t}\cdot\sin 2\pi nt\cdot dt+C,$$

or $$y=\frac{g}{b}\epsilon^{-\frac{a}{b}t}\int\epsilon^{\frac{a}{b}t}\cdot\sin 2\pi nt\cdot dt+C\epsilon^{-\frac{a}{b}t}. \qquad \text{[A]}$$

The last term is obviously a term which will die out as t increases, and may be omitted. The trouble now comes in to find the integral that appears as a factor. To tackle this we resort to the device (see p. 226) of integration by parts, the general formula for which is $\int u\,dv=uv-\int v\,du$. For this purpose write

$$\begin{cases} u=\epsilon^{\frac{a}{b}t}; \\ dv=\sin 2\pi nt\cdot dt. \end{cases}$$

We shall then have

$$\begin{cases} du=\epsilon^{\frac{a}{b}t}\times\frac{a}{b}dt; \\ v=-\frac{1}{2\pi n}\cos 2\pi nt. \end{cases}$$

Inserting these, the integral in question becomes:

$$\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt$$

$$= -\frac{1}{2\pi n} \cdot \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt - \int -\frac{1}{2\pi n} \cos 2\pi nt \cdot \epsilon^{\frac{a}{b}t} \cdot \frac{a}{b} dt$$

$$= -\frac{1}{2\pi n} \epsilon^{\frac{a}{b}t} \cos 2\pi nt + \frac{a}{2\pi nb} \int \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt \cdot dt. \ldots [\text{B}]$$

The last integral is still irreducible. To evade the difficulty, repeat the integration by parts of the left side, but treating it in the reverse way by writing:

$$\begin{cases} u = \sin 2\pi nt; \\ dv = \epsilon^{\frac{a}{b}t} \cdot dt; \end{cases}$$

whence
$$\begin{cases} du = 2\pi n \cdot \cos 2\pi nt \cdot dt; \\ v = \frac{b}{a} \epsilon^{\frac{a}{b}t} \end{cases}$$

Inserting these, we get

$$\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt$$

$$= \frac{b}{a} \cdot \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt - \frac{2\pi nb}{a} \int \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt \cdot dt. \ldots [\text{C}]$$

Noting that the final intractable integral in [C] is the same as that in [B], we may eliminate it by multiplying [B] by $\frac{2\pi nb}{a}$, and multiplying [C] by $\frac{a}{2\pi nb}$, and adding them.

The result, when cleared down, is:

$$\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt$$

$$= \epsilon^{\frac{a}{b}t} \left\{ \frac{ab \cdot \sin 2\pi nt - 2\pi nb^2 \cdot \cos 2\pi nt}{a^2 + 4\pi^2 n^2 b^2} \right\} \ldots\ldots [\text{D}]$$

Inserting this value in [A], we get

$$y = g \left\{ \frac{a \cdot \sin 2\pi nt - 2\pi nb \cdot \cos 2\pi nt}{a^2 + 4\pi^2 n^2 b^2} \right\}.$$

To simplify still further, let us imagine an angle ϕ such that $\tan \phi = \frac{2\pi nb}{a}$.

Then $\sin \phi = \frac{2\pi nb}{\sqrt{a^2 + 4\pi^2 n^2 b^2}}$,

and $\cos \phi = \frac{a}{\sqrt{a^2 + 4\pi^2 n^2 b^2}}$.

Substituting these, we get:

$$y = g \frac{\cos \phi \cdot \sin 2\pi nt - \sin \phi \cdot \cos 2\pi nt}{\sqrt{a^2 + 4\pi^2 n^2 b^2}},$$

which may be written

$$y = g \frac{\sin (2\pi nt - \phi)}{\sqrt{a^2 + 4\pi^2 n^2 b^2}},$$

which is *the solution* desired.

This is indeed none other than the equation of an alternating electric current, where g represents the amplitude of the electromotive force, n the frequency, a the resistance, b the coefficient of self-induction of the circuit, and ϕ is an angle of lag.

Example 4.

Suppose that $$Mdx+Ndy=0.$$

We could integrate this expression directly, if M were a function of x only, and N a function of y only; but, if both M and N are functions that depend on both x and y, how are we to integrate it? Is it itself an exact differential? That is: have M and N each been formed by partial differentiation from some common function U, or not? If they have, then

$$\begin{cases} \dfrac{\partial U}{\partial x}=M, \\ \dfrac{\partial U}{\partial y}=N. \end{cases}$$

And if such a common function exists, then

$$\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy$$

is an exact differential (compare p. 175).

Now the test of the matter is this. If the expression is an exact differential, it must be true that

$$\frac{dM}{dy}=\frac{dN}{dx};$$

for then $$\frac{d(dU)}{dx\,dy}=\frac{d(dU)}{dy\,dx},$$

which is necessarily true.

Take as an illustration the equation

$$(1+3xy)dx+x^2dy=0.$$

Is this an exact differential or not? Apply the test.

$$\begin{cases} \dfrac{d(1+3xy)}{dy}=3x, \\ \dfrac{d(x^2)}{dx} \quad =2x, \end{cases}$$

which do not agree. Therefore, it is not an exact differential, and the two functions $1+3xy$ and x^2 have not come from a common original function.

It is possible in such cases to discover, however, *an integrating factor*, that is to say, a factor such that if both are multiplied by this factor, the expression will become an exact differential. There is no one rule for discovering such an integrating factor; but experience will usually suggest one. In the present instance $2x$ will act as such. Multiplying by $2x$, we get

$$(2x+6x^2y)dx+2x^3dy=0.$$

Now apply the test to this.

$$\begin{cases} \dfrac{d(2x+6x^2y)}{dy}=6x^2, \\ \dfrac{d(2x^3)}{dx} \quad =6x^2, \end{cases}$$

which agrees. Hence this is an exact differential, and may be integrated. Now, if $w=2x^3y$,

$$dw=6x^2y\,dx+2x^3\,dy.$$

Hence $\int 6x^2y\,dx+\int 2x^3\,dy=w=2x^3y;$

so that we get $U=x^2+2x^3y+C.$

Example 5. Let $\frac{d^2y}{dt^2}+n^2y=0$.

In this case we have a differential equation of the second degree, in which y appears in the form of a second differential coefficient, as well as in person. Transposing, we have $\frac{d^2y}{dt^2}=-n^2y$.

It appears from this that we have to do with a function such that its second differential coefficient is proportional to itself, but with reversed sign. In Chapter XV. we found that there was such a function—namely, the *sine* (or the *cosine* also) which possessed this property. So, without further ado, we may infer that the solution will be of the form $y=A\sin(pt+q)$. However, let us go to work.

Multiply both sides of the original equation by $2\frac{dy}{dt}$ and integrate, giving us $2\frac{d^2y}{dt^2}\frac{dy}{dt}+2n^2y\frac{dy}{dt}=0$, and, as

$$2\frac{d^2y}{dt^2}\frac{dy}{dt}=\frac{d\left(\frac{dy}{dt}\right)^2}{dt},\quad \left(\frac{dy}{dt}\right)^2+n^2(y^2-C^2)=0,$$

C being a constant. Then, taking the square roots,

$$\frac{dy}{dt}=n\sqrt{C^2-y^2}\quad\text{and}\quad\frac{dy}{\sqrt{C^2-y^2}}=n\cdot dt.$$

But it can be shown that (see p. 171)

$$\frac{1}{\sqrt{C^2-y^2}}=\frac{d\left(\arcsin\frac{y}{C}\right)}{dy};$$

whence, passing from angles to sines,

$$\arcsin\frac{y}{C}=nt+C_1\quad\text{and}\quad y=C\sin(nt+C_1),$$

where C_1 is a constant angle that comes in by integration.

Or, preferably, this may be written

$y = A\sin nt + B\cos nt$, which is the solution.

Example 6. $\frac{d^2y}{dx^2} - n^2y = 0.$

Here we have obviously to deal with a function y which is such that its second differential coefficient is proportional to itself. The only function we know that has this property is the exponential function (see p. 143), and we may be certain therefore that the solution of the equation will be of that form.

Proceeding as before, by multiplying through by $2\frac{dy}{dx}$, and integrating, we get $2\frac{d^2y}{dx^2}\frac{dy}{dx} - 2n^2y\frac{dy}{dx} = 0$,

and, as $2\frac{d^2y}{dx^2}\frac{dy}{dx} = \frac{d\left(\frac{dy}{dx}\right)^2}{dx}$, $\left(\frac{dy}{dx}\right)^2 - n^2(y^2+c^2) = 0$,

$$\frac{dy}{dx} - n\sqrt{y^2+c^2} = 0,$$

where c is a constant, and $\frac{dy}{\sqrt{y^2+c^2}} = n\,dx$.

Now, if $w = \log_\epsilon(y+\sqrt{y^2+c^2}) = \log_\epsilon u$,

$$\frac{dw}{du} = \frac{1}{u},\quad \frac{du}{dy} = 1 + \frac{y}{\sqrt{y^2+c^2}} = \frac{y+\sqrt{y^2+c^2}}{\sqrt{y^2+c^2}}$$

and $$\frac{dw}{dy} = \frac{1}{\sqrt{y^2+c^2}}.$$

Hence, integrating, this gives us

$$\log_\epsilon(y+\sqrt{y^2+c^2}) = nx + \log_\epsilon C,$$

$$y+\sqrt{y^2+c^2}) = C\epsilon^{nx}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now $(y+\sqrt{y^2+c^2})\times(-y+\sqrt{y^2+c^2}) = c^2$;

whence $$-y+\sqrt{y^2+c^2} = \frac{c^2}{C}\epsilon^{-nx}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting (2) from (1) and dividing by 2, we then have

$$y = \frac{1}{2} C\epsilon^{nx} - \frac{1}{2}\frac{c^2}{C}\epsilon^{-nx},$$

which is more conveniently written

$$y = A\epsilon^{nx} + B\epsilon^{-nx}.$$

Or, the solution, which at first sight does not look as if it had anything to do with the original equation, shows that y consists of two terms, one of which grows logarithmically as x increases, while the other term dies away as x increases.

Example 7.

Let $$b\frac{d^2y}{dt^2} + a\frac{dy}{dt} + gy = 0.$$

Examination of this expression will show that, if $b = 0$, it has the form of Example 1, the solution of which was a negative exponential. On the other hand, if $a = 0$, its form becomes the same as that of Example 6, the solution of which is the sum of a positive and a negative exponential. It is therefore not very surprising to find that the solution of the present example is

$$y = (\epsilon^{-mt})(A\epsilon^{nt} + B\epsilon^{-nt}),$$

where $$m = \frac{a}{2b} \quad \text{and} \quad n = \sqrt{\frac{a^2}{4b^2} - \frac{g}{b}}.$$

The steps by which this solution is reached are not given here; they may be found in advanced treatises.

Example 8.

$$\frac{d^2y}{dt^2}=a^2\frac{d^2y}{dx^2}.$$

It was seen (p. 177) that this equation was derived from the original

$$y=F(x+at)+f(x-at),$$

where F and f were any arbitrary functions of t.

Another way of dealing with it is to transform it by a change of variables into

$$\frac{d^2y}{du\cdot dv}=0,$$

where $u=x+at$, and $v=x-at$, leading to the same general solution. If we consider a case in which F vanishes, then we have simply

$$y=f(x-at);$$

and this merely states that, at the time $t=0$, y is a particular function of x, and may be looked upon as denoting that the curve of the relation of y to x has a particular shape. Then any change in the value of t is equivalent simply to an alteration in the origin from which x is reckoned. That is to say, it indicates that, the form of the function being conserved, it is propagated along the x direction with a uniform velocity a; so that whatever the value of the ordinate y at any particular time t_0 at any particular point x_0, the same value of y will appear at the subsequent time t_1 at a point further along, the abscissa of which is $x_0+a(t_1-t_0)$. In this case the simplified equation represents the propagation of a wave (of any form) at a uniform speed along the x direction.

If the differential equation had been written

$$m\frac{d^2y}{dt^2}=k\frac{d^2y}{dx^2},$$

the solution would have been the same, but the velocity of propagation would have had the value

$$a=\sqrt{\frac{k}{m}}.$$

CHAPTER XXII.

A LITTLE MORE ABOUT CURVATURE OF CURVES.

In Chapter XII. we have learned how we can find out which way a curve is curved, that is, whether it curves upwards or downwards towards the right. This gave us no indication whatever as to *how much* the curve is curved, or, in other words, what is its *curvature.*

By *curvature* of a line, we mean the amount of bending or deflection taking place along a certain length of the line, say along a portion of the line the length of which is one unit of length (the same unit which is used to measure the radius, whether it be one inch, one foot, or any other unit). For instance, consider two circular paths of centre O and O' and of equal lengths AB, $A'B'$ (see Fig. 64). When passing from A to B along the arc AB of the first one, one changes one's direction from AP to BQ, since at A one faces in the direction AP and at B one faces in the direction BQ. In other words, in walking from A to B one unconsciously turns round through the angle PCQ, which is equal to the angle AOB. Similarly, in passing from A' to B', along the arc $A'B'$, of equal length to AB, on the second path, one turns round through the angle $P'C'Q'$, which is equal to the angle $A'O'B'$, obviously *greater* than the correspond-

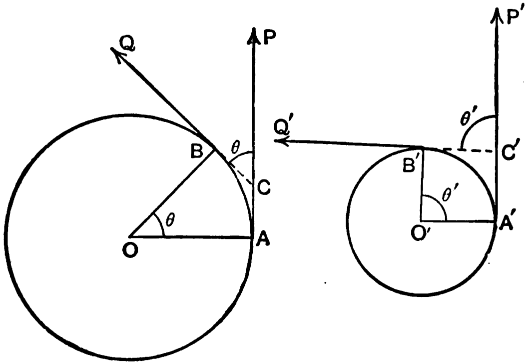


FIG. 64.

ing angle AOB. The second path bends therefore more than the first for an equal length.

This fact is expressed by saying that the *curvature* of the second path is greater than that of the first one. The larger the circle, the lesser the bending, that is the lesser the curvature. If the radius of the first circle is 2, 3, 4, ... etc. times greater than the radius of the second, then the angle of bending or deflection along an arc of unit length will be 2, 3, 4, ... etc. times less on the first circle than on the second, that is, it will be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... etc. of the bending or deflection along the arc of same length on the second circle. In other words, the *curvature* of the first circle will be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... etc. of that of the second circle. We see that, as the radius becomes 2, 3, 4, .. etc. times greater, the curvature becomes 2, 3, 4, ... etc. times smaller, and this is expressed by saying that *the curvature of a circle is inversely proportional to the radius of the circle*, or

$$\text{curvature} = k \times \frac{1}{\text{radius}},$$

where k is a constant. It is agreed to take $k = 1$, so that

$$\text{curvature} = \frac{1}{\text{radius}},$$

always.

If the radius becomes indefinitely great, the curvature becomes $\frac{1}{\text{infinity}} = \text{zero}$, since when the denominator of a fraction is indefinitely large, the value of the fraction is indefinitely small. For this reason mathematicians sometimes consider a straight line as an arc of circle of infinite radius, or zero curvature.

In the case of a circle, which is perfectly symmetrical and uniform, so that the curvature is the same at every point of its circumference, the above method of expressing the curvature is perfectly definite. In the case of any other curve, however, the curvature is not the same at different points, and it may differ considerably even for two points fairly close to one another. It would not then be accurate to take the amount of bending or deflection between two points as a measure of the curvature of the arc between these points, unless this arc is very small, in fact, unless it is indefinitely small.

If then we consider a very small arc such as AB (see Fig. 65), and if we draw such a circle that an arc AB

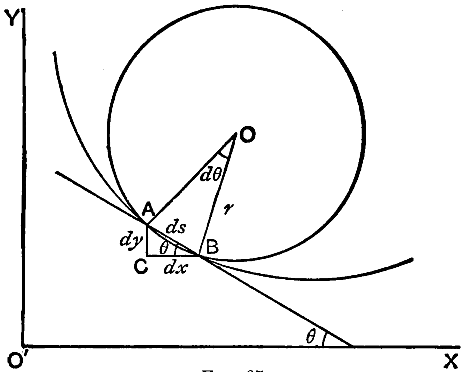


FIG. 65.

of this circle coincides with the arc AB of the curve more closely than would be the case with any other circle, then the curvature of this circle may be taken as the curvature of the arc AB of the curve. The smaller the arc AB, the easier it will be to find a circle an arc of which most nearly coincides with the arc AB of the curve. When A and B are very near one another, so that AB is so small so that the length ds of the arc AB is practically negligible, then the coincidence of the two arcs, of circle and of curve, may be considered as being practically perfect, and the curvature of the curve at the point A (or B), being then the same as the curvature of the circle, will be expressed by the reciprocal of the radius of this circle, that is, by $\frac{1}{OA}$, according to our way of measuring curvature, explained above.

Now, at first, you may think that, if AB is very small, then the circle must be very small also. A little thinking will, however, cause you to perceive that it is by no means necessarily so, and that the circle may have any size, according to the amount of bending of the curve along this very small arc AB. In fact, if the curve is almost flat at that point, the circle will be extremely large. This circle is called the *circle of curvature*, or the *osculating circle* at the point considered. Its radius is the *radius of curvature* of the curve at that particular point.

If the arc AB is represented by ds and the angle AOB by $d\theta$, then, if r is the radius of curvature,

$$ds = r\,d\theta \quad \text{or} \quad \frac{d\theta}{ds} = \frac{1}{r}.$$

The secant AB makes with the axis OX the angle θ, and it will be seen from the small triangle ABC that $\frac{dy}{dx} = \tan\theta$. When AB is indefinitely small, so that B practically coincides with A, the line AB becomes a tangent to the curve at the point A (or B).

Now, $\tan\theta$ depends on the position of the point A (or B, which is supposed to nearly coincide with it), that is, it depends on x, or, in other words, $\tan\theta$ is "a function" of x.

Differentiating with regard to x to get the slope (see p. 112), we get

$$\frac{d\left(\frac{dy}{dx}\right)}{dx}=\frac{d(\tan\theta)}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2}=\sec^2\theta\,\frac{d\theta}{dx}=\frac{1}{\cos^2\theta}\,\frac{d\theta}{dx}$$

(see p. 168);

hence $$\frac{d\theta}{dx}=\cos^2\theta\,\frac{d^2y}{xd^2}.$$

But $\frac{dx}{ds}=\cos\theta$, and for $\frac{d\theta}{ds}$ one may write $\frac{d\theta}{dx}\times\frac{dx}{ds}$; therefore

$$\frac{1}{r}=\frac{d\theta}{ds}=\frac{d\theta}{dx}\times\frac{dx}{ds}=\cos^3\theta\,\frac{d^2y}{dx^2}=\frac{\frac{d^2y}{dx^2}}{\sec^3\theta};$$

but $\sec\theta=\sqrt{1+\tan^2\theta}$; hence

$$\frac{1}{r}=\frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1+\tan^2\theta}\right)^3}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}};$$

and finally, $$r=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

The numerator, being a square root, may have the sign + or the sign −. One must select for it the same sign as the denominator, so as to have r positive always, as a negative radius would have no meaning.

It has been shown (Chapter XII.), that if $\frac{d^2y}{dx^2}$ is positive, the curve is convex downwards, while if $\frac{d^2y}{dx^2}$ is negative, the curve is concave downwards. If $\frac{d^2y}{dx^2}=0$, the radius of curvature is infinitely great, that is, the corresponding portion of the curve is a bit of straight line. This necessarily happens whenever a curve gradually changes from being convex to concave to the axis of x or vice versa. The point where this occurs is called a *point of inflexion.*

The centre of the circle of curvature is called the *centre of curvature.* If its coordinates are x_1, y_1, then the equation of the circle is (see p. 102)

$$(x-x_1)^2+(y-y_1)^2=r^2;$$

hence $$2(x-x_1)dx+2(y-y_1)dy=0,$$

and $$x-x_1+(y-y_1)\frac{dy}{dx}=0. \quad\text{............(1)}$$

Why did we differentiate? To get rid of the constant r. This leaves but two unknown constants x_1 and y_1; differentiate again; you shall get rid of one of them. This last differentiation is not quite as easy as it seems; let us do it together; we have:

$$\frac{d(x)}{dx}+\frac{d\left[(y-y_1)\frac{dy}{dx}\right]}{dx}=0;$$

the numerator of the second term is a product; hence differentiating it gives

$$(y-y_1)\frac{d\left(\frac{dy}{dx}\right)}{dx}+\frac{dy}{dx}\frac{d(y-y_1)}{dx}=(y-y_1)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2,$$

so that the result of differentiating (1) is

$$1+\left(\frac{dy}{dx}\right)^2+(y-y_1)\frac{d^2y}{dx^2}=0;$$

from this we at once get

$$y_1=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

Replacing in (1), we get

$$(x-x_1)+\left\{y-y-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right\}\frac{dy}{dx}=0;$$

and finally, $$x_1=x-\frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}};$$

x_1 and y_1 give the position of the centre of curvature. The use of these formulae will be best seen by carefully going through a few worked-out examples.

Example 1. Find the radius of curvature and the coordinates of the centre of curvature of the curve $y=2x^2-x+3$ at the point $x=0$.

We have $\frac{dy}{dx}=4x-1, \quad \frac{d^2y}{dx^2}=4.$

$$r=\frac{\pm\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\{1+(4x-1)^2\}^{\frac{3}{2}}}{4},$$

when $x=0$; this becomes

$$\frac{\{1+(-1)^2\}^{\frac{3}{2}}}{4}=\frac{\sqrt{8}}{4}=0{\cdot}707$$

If x_1, y_1 are the coordinates of the centre of curvature then

$$x_1=x-\frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}=x-\frac{(4x-1)\{1+(4x-1)^2\}}{4}$$

$$=0-\frac{(-1)\{1+(-1)^2\}}{4}=\frac{1}{2}$$

when $x=0$, $y=3$, so that

$$y_1=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}=y+\frac{1+(4x-1)^2}{4}=3+\frac{1+(-1)^2}{4}=3\tfrac{1}{2}.$$

Plot the curve and draw the circle, it is both interesting and instructive. The values can be checked easily, as since when $x=0$, $y=3$, here

$$x_1{}^2+(y_1-3)^2=r^2 \quad \text{or} \quad {\cdot}5^2+{\cdot}5^2={\cdot}50={\cdot}707^2.$$

Example 2. Find the radius of curvature and the position of the centre of curvature of the curve $y^2=mx$ at the point for which $y=0$.

Here $$y=m^{\frac{1}{2}}x^{\frac{1}{2}},\quad \frac{dy}{dx}=\frac{1}{2}m^{\frac{1}{2}}x^{-\frac{1}{2}}=\frac{m^{\frac{1}{2}}}{2x^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2}\times\frac{m^{\frac{1}{2}}}{2}x^{-\frac{3}{2}}=-\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}};$$

hence

$$\frac{\pm\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\pm\left\{1+\frac{m}{4x}\right\}^{\frac{3}{2}}}{-\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}}=\frac{(4x+m)^{\frac{3}{2}}}{2m^{\frac{1}{2}}},$$

taking the $-$ sign at the numerator, so as to have r positive.

Since, when $y=0$, $x=0$, we get $r=\dfrac{m^{\frac{3}{2}}}{2m^{\frac{1}{2}}}=\dfrac{m}{2}$.

Also, if x_1, y_1 are the coordinates of the centre,

$$x_1=x-\frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{d^2x}}=x-\frac{\frac{m^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\left\{1+\frac{m}{4x}\right\}}{-\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}}$$

$$=x+\frac{4x+m}{2}=3x+\frac{m}{2},$$

when $x=0$, then $x_1=\dfrac{m}{2}$.

Also

$$y_1 = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = m^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{1+\frac{m}{4x}}{\frac{m^{\frac{1}{2}}}{4x^{\frac{3}{2}}}} = -\frac{4x^{\frac{5}{2}}}{m^{\frac{1}{2}}};$$

when $x=0$, $y_1=0$.

Example 3. Show that the circle is a curve of constant curvature.

If x_1, y_1 are the coordinates of the centre, and R is the radius, the equation of the circle in rectangular coordinates is

$$(x-x_1)^2+(y-y_1)^2=R^2;$$

this is easily put into the form

$$y=\sqrt{R^2-(x-x_1)^2}+y_1=\{R^2-(x-x_1)^2\}^{\frac{1}{2}}+y_1.$$

To differentiate, let $R^2-(x-x_1)^2=v$; then

$$y=v^{\frac{1}{2}}+y_1, \quad \frac{dy}{dv}=\tfrac{1}{2}v^{-\frac{1}{2}}, \quad \frac{dv}{dx}=-2(x-x_1),$$

$$\frac{dy}{dx}=\frac{dy}{dv}\times\frac{dv}{dx}=-\tfrac{1}{2}\{R^2-(x-x_1)^2\}^{-\frac{1}{2}}\times 2(x-x_1)$$

$$=\frac{-(x-x_1)}{\{R^2-(x-x_1)^2\}^{\frac{1}{2}}}.$$

Differentiate again; using the rule for differentiation of a fraction, we get

$$\frac{d^2y}{dx^2}=\frac{\{R^2-(x-x_1)^2\}^{\frac{1}{2}}\times\frac{d}{dx}\{-(x-x_1)\}-\{-(x-x_1)\}\times\frac{d}{dx}\{R^2-(x-x_1)^2\}^{\frac{1}{2}}}{R^2-(x-x_1)^2}$$

(it is always a good plan to write out the whole expression in this way when dealing with a complicated expression); this simplifies to

$$\frac{d^2y}{dx^2}=\frac{\{R^2-(x-x_1)^2\}^{\frac{1}{2}}(-1)-\dfrac{(x-x_1)^2}{\{R^2-(x-x_1)^2\}^{\frac{1}{2}}}}{R^2-(x-x_1)^2}$$

$$=\frac{R^2}{\{R^2-(x-x_1)^2\}^{\frac{3}{2}}};$$

hence

$$r=\frac{\pm\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=\frac{\left\{1+\dfrac{(x-x_1)^2}{R^2-(x-x_1)^2}\right\}^{\frac{3}{2}}}{\dfrac{R^2}{\{R^2-(x-x_1)^2\}^{\frac{3}{2}}}}=\frac{(R^2)^{\frac{3}{2}}}{R^2}=R;$$

the radius of curvature is constant and equal to the radius of the circle.

Example 4. Find the radius and the centre of curvature of the curve $y=x^3-2x^2+x-1$ at points where $x=0$, $x=0{\cdot}5$ and $x=1{\cdot}0$. Find also the position of the point of inflexion of the curve.

Here $\dfrac{dy}{dx}=3x^2-4x+1$, $\dfrac{d^2y}{dx^2}=6x-4$.

$$r=\frac{\{1+(3x^2-4x+1)^2\}^{\frac{3}{2}}}{6x-4},$$

$$x_1=x-\frac{(3x^2-4x+1)\{1+(3x^2-4x+1)^2\}}{6x-4},$$

$$y_1=y+\frac{1+(3x^2-4x+1)^2}{6x-4}.$$

When $x=0$, $y=-1$,

$$r=\frac{\sqrt{8}}{4}=0{\cdot}707. \quad x_1=0+\tfrac{1}{2}=0{\cdot}5, \quad y_1=-1-\tfrac{1}{2}=-1{\cdot}5.$$

Plot the curve, mark the point $x=0$, $y=-1$, take two points on either side about half an inch away and construct geometrically the circle passing by the three points; measure the radius and the coordinates of the centre, and compare with the above results. On a diagram, the scale of which was 2 inches = unit length, the construction gave a circle for which $r=0{\cdot}72$, $x_1=0{\cdot}47$, $y_1=-1{\cdot}53$, a very fair agreement.

When $x=0{\cdot}5$, $y=-0{\cdot}875$,

$$r=\frac{-\{1+(-0{\cdot}25)^2\}^{\frac{3}{2}}}{-1}=1{\cdot}09,$$

$$x_1=0{\cdot}5-\frac{-0{\cdot}25\times1{\cdot}09}{-1}=0{\cdot}33,$$

$$y_1=-0{\cdot}875+\frac{1{\cdot}09}{-1}=-1{\cdot}96.$$

The diagram gave $r=0{\cdot}98$, $x_1=0{\cdot}33$, $y_1=-1{\cdot}83$.

When $x=1$, $y=-1$,

$$r=\frac{(1+0)^{\frac{3}{2}}}{2}=0{\cdot}5,$$

$$x_1=1-\frac{0\times(1+0)}{2}=1,$$

$$y_1=-1+\frac{1+0^2}{2}=-0{\cdot}5.$$

The diagram gave $r=0{\cdot}57$, $x_1=0{\cdot}96$, $y_1=-0{\cdot}44$.

At the point of inflexion $\frac{d^2y}{dx^2}=0$, $6x-4=0$, and $x=\frac{2}{3}$; hence $y=0{\cdot}925$.

Example 5. Find the radius and centre of curvature of the curve $y=\frac{a}{2}\left\{\epsilon^{\frac{x}{a}}+\epsilon^{-\frac{x}{a}}\right\}$, at the point for which $x=0$. (This curve is called the *catenary*, as a hanging chain affects the same slope exactly.) The equation of the curve may be written

$$y=\frac{a}{2}\epsilon^{\frac{x}{a}}+\frac{a}{2}\epsilon^{-\frac{x}{a}};$$

then (see p. 150 Examples),

$$\frac{dy}{dx}=\frac{a}{2}\times\frac{1}{a}\epsilon^{\frac{x}{a}}-\frac{a}{2}\times\frac{1}{a}\epsilon^{-\frac{x}{a}}=\frac{1}{2}\left(\epsilon^{\frac{x}{a}}-\epsilon^{-\frac{x}{a}}\right).$$

Similarly

$$\frac{d^2y}{dx^2}=\frac{1}{2a}\left\{\epsilon^{\frac{x}{a}}+\epsilon^{-\frac{x}{a}}\right\}=\frac{1}{2a}\times\frac{2y}{a}=\frac{y}{a^2},$$

$$r=\frac{\left\{1+\frac{1}{4}\left(\epsilon^{\frac{x}{a}}-\epsilon^{-\frac{x}{a}}\right)^2\right\}^{\frac{3}{2}}}{\frac{y}{a^2}}=\frac{a^2}{8y}\sqrt{\left(2+\epsilon^{\frac{2x}{a}}+\epsilon^{-\frac{2x}{a}}\right)^3},$$

since $\epsilon^{\frac{x}{a}-\frac{x}{a}}=\epsilon^0=1$, or

$$r=\frac{a^2}{8y}\sqrt{\left(2\epsilon^{\frac{x}{a}-\frac{x}{a}}+\epsilon^{\frac{2x}{a}}+\epsilon^{-\frac{2x}{a}}\right)^3}=\frac{a^2}{8y}\sqrt{\left(\epsilon^{\frac{x}{a}}+\epsilon^{-\frac{x}{a}}\right)^6}=\frac{y^2}{a},$$

when $x=0$, $y=\frac{a}{2}(\epsilon^0+\epsilon^0)=a$, $\frac{dy}{dx}=\frac{1}{2}(\epsilon^0-\epsilon^0)=0$;

hence $r=\frac{a^2}{a}=a$.

The radius of curvature at the vertex is equal to the constant a.

Also $$x_1=0-\frac{0(1+0)}{\frac{1}{a}}=0,$$

$$y_1=y+\frac{1+0}{\frac{1}{a}}=a+a=2a.$$

You are now sufficiently familiar with this type of problem to work out the following exercises by yourself. You are advised to check your answers by careful plotting of the curve and construction of the circle of curvature, as explained in Example 4.

Exercises XX. (For Answers see p. 299.)

(1) Find the radius of curvature and the position of the centre of curvature of the curve $y=\epsilon^x$ at the point for which $x=0$.

(2) Find the radius and the centre of curvature of the curve $y=x\left(\frac{x}{2}-1\right)$ at the point for which $x=2$.

(3) Find the point or points of curvature unity in the curve $y=x^2$.

(4) Find the radius and the centre of curvature of the curve $xy=m$, at the point for which $x=\sqrt{m}$.

(5) Find the radius and the centre of curvature of the curve $y^2=4ax$ at the point for which $x=0$.

(6) Find the radius and the centre of curvature of

the curve $y=x^3$ at the points for which $x=\pm 0{\cdot}9$ and also $x=0$.

(7) Find the radius of curvature and the coordinates of the centre of curvature of the curve

$$y=x^2-x+2$$

at the two points for which $x=0$ and $x=1$, respectively. Find also the maximum or minimum value of y. Verify graphically all your results.

(8) Find the radius of curvature and the coordinates of the centre of curvature of the curve

$$y=x^3-x-1$$

at the points for which $x=-2$, $x=0$, and $x=1$.

(9) Find the coordinates of the point or points of inflexion of the curve $y=x^3+x^2+1$.

(10) Find the radius of curvature and the coordinates of the centre of curvature of the curve

$$y=(4x-x^2-3)^{\frac{1}{2}}$$

at the points for which $x=1{\cdot}2$, $x=2$ and $x=2{\cdot}5$. What is this curve?

(11) Find the radius and the centre of curvature of the curve $y=x^3-3x^2+2x+1$ at the points for which $x=0$, $x=+1{\cdot}5$. Find also the position of the point of inflexion.

(12) Find the radius and centre of curvature of the curve $y=\sin\theta$ at the points for which $\theta=\frac{\pi}{4}$ and $\theta=\frac{\pi}{2}$. Find the position of the point of inflexion.

(13) Draw a circle of radius 3, the centre of which has for its coordinates $x=1$, $y=0$. Deduce the equation of such a circle from first principles (see p. 102). Find by calculation the radius of curvature and the coordinates of the centre of curvature for several suitable points, as accurately as possible, and verify that you get the known values.

(14) Find the radius and centre of curvature of the curve $y=\cos\theta$ at the points for which $\theta=0$, $\theta=\frac{\pi}{4}$ and $\theta=\frac{\pi}{2}$.

(15) Find the radius of curvature and the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points for which $x=0$ and at the points for which $y=0$.

CHAPTER XXIII.

HOW TO FIND THE LENGTH OF AN ARC ON A CURVE.

SINCE an arc on any curve is made up of a lot of little bits of straight lines joined end to end, if we could add all these little bits, we would get the length of the arc. But we have seen that to add a lot of little bits together is precisely what is called

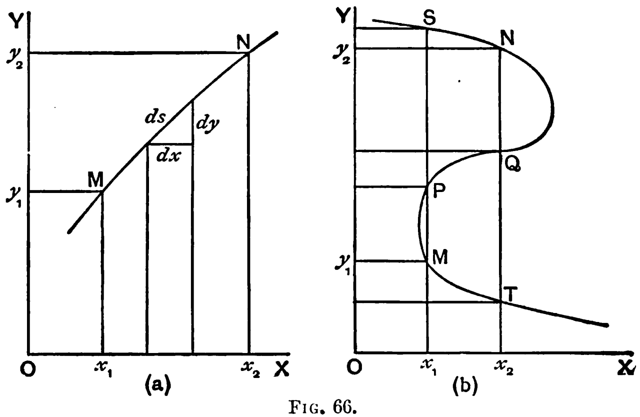


FIG. 66.

integration, so that it is likely that, since we know how to integrate, we can find also the length of an arc on any curve, provided that the equation of the curve is such that it lend itself to integration.

If MN is an arc on any curve, the length s of which is required (see Fig. 66a), if we call "a little bit" of the arc ds, then we see at once that

$$(ds)^2 = (dx)^2 + (dy)^2,$$

or either

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Now the arc MN is made up of the sum of all the little bits ds between M and N, that is, between x_1 and x_2, or between y_1 and y_2, so that we get either

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

That is all!

The second integral is useful when there are several points of the curve corresponding to the given values of x (as in Fig 66b). In this case the integral between x_1 and x_2 leaves a doubt as to the exact portion of the curve, the length of which is required. It may be ST, instead of MN, or SQ; by integrating between y_1 and y_2 the uncertainty is removed, and in this case one should use the second integral.

If instead of x and y coordinates,—or Cartesian coordinates, as they are named from the French mathematician Descartes, who invented them—we have r and θ coordinates (or polar coordinates, see p. 219); then, if MN be a small arc of length ds on any curve, the length s of which is required (see Fig. 67), O being the pole, then the distance ON will generally differ from OM by a small amount dr. If the small angle MON is called $d\theta$, then, the polar coordinates of the point M being θ and r, those of N are $(\theta+d\theta)$ and $(r+dr)$. Let MP be perpendicular

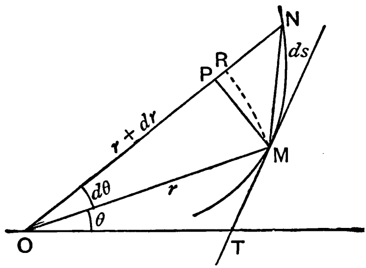


FIG. 67.

to ON, and let $OR=OM$; then $RN=dr$, and this is very nearly the same as PN, as long as $d\theta$ is a very small angle. Also $RM=r\,d\theta$, and RM is very nearly equal to PM, and the arc MN is very nearly equal to the chord MN. In fact we can write $PN=dr$, $PM=r\,d\theta$, and arc $MN=$ chord MN without appreciable error, so that we have.

$$(ds)^2=(\text{chord } MN)^2=\overline{PN}^2+\overline{PM}^2=dr^2+r^2d\theta^2.$$

Dividing by $d\theta^2$ we get $\left(\dfrac{ds}{d\theta}\right)^2=r^2+\left(\dfrac{dr}{d\theta}\right)^2$; hence

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \quad\text{and}\quad ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta;$$

hence, since the length s is made up of the sum of all the little bits ds, between values of $\theta=\theta_1$ and $\theta=\theta_2$, we have

$$s=\int_{\theta_1}^{\theta_2} ds=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

We can proceed at once to work out a few examples.

Example 1. The equation of a circle, the centre of which is at the origin—or intersection of the axis of x with the axis of y—is $x^2+y^2=r^2$; find the length of an arc of one quadrant.

$y^2=r^2-x^2$ and $2y\,dy=-2x\,dx$, so that $\dfrac{dy}{dx}=-\dfrac{x}{y}$; hence

$$s=\int\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx=\int\sqrt{\left(1+\frac{x^2}{y^2}\right)}\,dx;$$

and since $y^2=r^2-x^2$,

$$s=\int\sqrt{\left(1+\frac{x^2}{r^2-x^2}\right)}\,dx=\int\frac{r\,dx}{\sqrt{(r^2-x^2)}}.$$

The length we want—one quadrant—extends from a point for which $x=0$ to another point for which $x=r$. We express this by writing

$$s=\int_{x=0}^{x=r}\frac{r\,dx}{\sqrt{(r^2-x^2)}},$$

or, more simply, by writing

$$s=\int_0^r\frac{r\,dx}{\sqrt{(r^2-x^2)}},$$

the 0 and r to the right of the sign of integration merely meaning that the integration is only to be performed on a portion of the curve, namely that between $x=0$, $x=r$, as we have seen (p. 210).

Here is a fresh integral for you! Can you manage it?

On p. 171 we have differentiated $y=\text{arc}(\sin x)$ and found $\frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}$. If you have tried all sorts of variations of the given examples (as you ought to have done!), you perhaps tried to differentiate something like $y=a\,\text{arc}\left(\sin\frac{x}{a}\right)$, which gave

$$\frac{dy}{dx}=\frac{a}{\sqrt{(a^2-x^2)}} \quad \text{or} \quad dy=\frac{a\,dx}{\sqrt{(a^2-x^2)}},$$

that is, just the same expression as the one we have to integrate here.

Hence $s=\int\frac{r\,dx}{\sqrt{(r^2-x^2)}}=r\,\text{arc}\left(\sin\frac{x}{r}\right)+C$, C being a constant.

As the integration is only to be made between $x=0$ and $x=r$, we write

$$s=\int_0^r\frac{r\,dx}{\sqrt{(r^2-x^2)}}=\left[r\,\text{arc}\left(\sin\frac{x}{r}\right)+C\right]_0^r;$$

proceeding then as explained in Example (1), p. 211, we get

$$s=r\,\text{arc}\left(\sin\frac{r}{r}\right)+C-r\,\text{arc}\left(\sin\frac{0}{r}\right)-C,$$

or

$$s=r\times\frac{\pi}{2},$$

since arc(sin 1) is 90° or $\frac{\pi}{2}$ and arc(sin 0) is zero, and the constant C disappears, as has been shown.

The length of the quadrant is therefore $\frac{\pi r}{2}$, and the length of the circumference, being four times this, is $4 \times \frac{\pi r}{2} = 2\pi r$.

Example 2. Find the length of the arc AB between $x_1 = 2$ and $x_2 = 5$, in the circumference $x^2 + y^2 = 6^2$ (see Fig. 68).

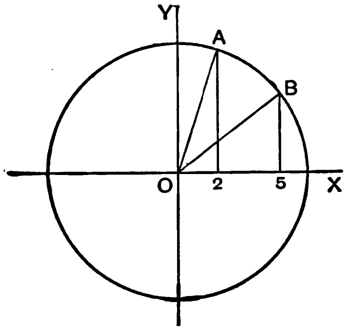


FIG. 68.

Here, proceeding as in previous example,

$$s = \Big[r \text{ arc sin} \left(\frac{x}{r}\right) + C \Big]_{x_1}^{x_2} = \Big[6 \text{ arc sin} \left(\frac{x}{6}\right) + C \Big]_2^5$$

$$= 6 \Big[\text{arc sin} \left(\frac{5}{6}\right) - \text{arc sin} \left(\frac{2}{6}\right) \Big] = 6\,(0{\cdot}9850 - 0{\cdot}3397)$$

$= 3{\cdot}8718$ inches (the arcs being expressed in radians).

It is always well to check results obtained by a new and yet unfamiliar method. This is easy, for

$$\cos AOX = \tfrac{2}{6} = \tfrac{1}{3} \text{ and } \cos BOX = \tfrac{5}{6};$$

hence $AOX = 70^\circ 32'$, $BOX = 33^\circ 34'$,

and $AOX - BOX = AOB = 36^\circ 58'$

$$= \frac{36 \cdot 9667}{57 \cdot 2958} \text{ radian} = 0 \cdot 6451 \text{ radian} = 3 \cdot 8706 \text{ inches},$$

the discrepancy being merely due to the fact that the last decimal in logarithmic and trigonometrical tables is only approximate.

Example 3. Find the length of an arc of the curve

$$y = \frac{a}{2}\left\{\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}}\right\}$$

between $x=0$ and $x=a$. (This curve is the *catenary*.)

$$y = \frac{a}{2}\epsilon^{\frac{x}{a}} + \frac{a}{2}\epsilon^{-\frac{x}{a}}, \quad \frac{dy}{dx} = \frac{1}{2}\left\{\epsilon^{\frac{x}{a}} - \epsilon^{-\frac{x}{a}}\right\},$$

$$s = \int \sqrt{1 + \frac{1}{4}\left\{\epsilon^{\frac{x}{a}} - \epsilon^{-\frac{x}{a}}\right\}^2}\, dx$$

$$= \frac{1}{2}\int \sqrt{4 + \epsilon^{\frac{2x}{a}} + \epsilon^{-\frac{2x}{a}} - 2\epsilon^{\frac{x}{a} - \frac{x}{a}}}\, dx.$$

Now

$$\epsilon^{\frac{x}{a} - \frac{x}{a}} = \epsilon^0 = 1, \text{ so that } s = \frac{1}{2}\int \sqrt{2 + \epsilon^{\frac{2x}{a}} + \epsilon^{-\frac{2x}{a}}}\, dx;$$

we can replace 2 by $2 \times \epsilon^0 = 2 \times \epsilon^{\frac{x}{a} - \frac{x}{a}}$; then

$$s=\frac{1}{2}\int\sqrt{\epsilon^{\frac{2x}{a}}+2\epsilon^{\frac{x}{a}-\frac{x}{a}}+\epsilon^{-\frac{2x}{a}}}\,dx$$

$$=\frac{1}{2}\int\sqrt{\left(\epsilon^{\frac{x}{a}}+\epsilon^{-\frac{x}{a}}\right)^2}\,dx=\frac{1}{2}\int\left(\epsilon^{\frac{x}{a}}+\epsilon^{-\frac{x}{a}}\right)dx$$

$$=\frac{1}{2}\int\epsilon^{\frac{x}{a}}\,dx+\frac{1}{2}\int\epsilon^{-\frac{x}{a}}\,dx=\frac{a}{2}\left[\epsilon^{\frac{x}{a}}-\epsilon^{-\frac{x}{a}}\right].$$

Here $s=\frac{a}{2}\left[\epsilon^{\frac{x}{a}}-\epsilon^{-\frac{x}{a}}\right]_0^a=\frac{a}{2}\left[\epsilon^1-\epsilon^{-1}+1-1\right],$

and $s=\frac{a}{2}\left(\epsilon-\frac{1}{\epsilon}\right).$

Example 4. A curve is such that the length of the tangent at any point P (see Fig. 69) from P to the

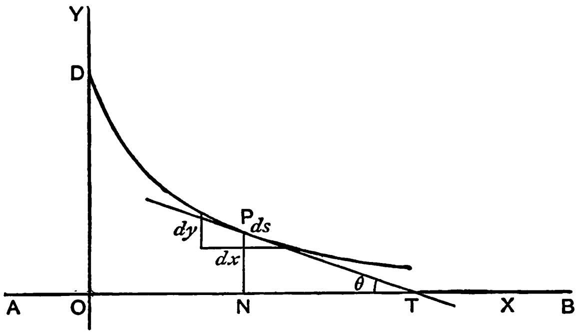


FIG. 69.

intersection T of the tangent with a fixed line AB is a constant length a. Find an expression for an arc of this curve,—which is called the tractrix,—and find the length, when $a=3$, between the ordinates $y=a$ and $y=1$.

We shall take the fixed line for the axis of x. The point D, with $DO=a$, is a point on the curve, which must be tangent to OD at D. We take OD as the axis of y; AB and OD are what are called axes of symmetry, that is the curve is symmetrical about them; $PT=a$, $PN=y$, $ON=x$.

If we consider a small portion ds of the curve, at P, then $\sin\theta=\frac{dy}{ds}=-\frac{y}{a}$ (minus because the curve slopes *downwards* to the right, see p. 79).

Hence $\frac{ds}{dy}=-\frac{a}{y}$, $ds=-a\frac{dy}{y}$ and $s=-a\int\frac{dy}{y}$, that is, $s=-a\log_\epsilon y+C$

When $x=0$, $s=0$, $y=a$, so that $0=-a\log_\epsilon a+C$, and $C=a\log_\epsilon a$.

It follows that $s=a\log_\epsilon a-a\log_\epsilon y=a\log_\epsilon\frac{a}{y}$.

When $a=3$, s between $y=a$ and $y=1$ is therefore

$$s=3\left[\log_\epsilon\frac{3}{y}\right]_1^3=3(\log_\epsilon 1-\log_\epsilon 3)=3\times(0-1{\cdot}0986)$$
$$=-3{\cdot}296 \text{ or } 3{\cdot}296,$$

as the sign $-$ refers merely to the direction in which the length was measured, from D to P, or from P to D.

Note that this result has been obtained without a knowledge of the equation of the curve. This is sometimes possible. In order to get the length of an arc between two points given by their abscissae, however, it is necessary to know the equation of the curve; this is easily obtained as follows:

$$\frac{dy}{dx} = -\tan\theta = -\frac{y}{\sqrt{a^2-y^2}}, \text{ since } PT = a;$$

hence
$$dx = -\frac{\sqrt{a^2-y^2}\,dy}{y}.$$

The integration will give us a relation between x and y, which is the equation of the curve

$$x = -\int\frac{\sqrt{a^2-y^2}\,dy}{y} = -a^2\int\frac{dy}{y\sqrt{a^2-y^2}} + \int\frac{y\,dy}{\sqrt{a^2-y^2}}.$$

To integrate

$$\frac{dy}{y\sqrt{a^2-y^2}} \quad \text{let } y = \frac{1}{z}; \quad \text{then } \frac{dy}{dz} = -\frac{1}{z^2} = -\frac{y}{z},$$

so that
$$\frac{dy}{y} = -\frac{dz}{z}.$$

The integral becomes $-\int\frac{dz}{\sqrt{a^2z^2-1}}$. To integrate this let $\sqrt{a^2z^2-1} = v - az$, that is,

$$a^2z^2 - 1 = v^2 + a^2z^2 - 2avz$$

and
$$0 = 2v\,dv - 2az\,dv - 2av\,dz,$$

from which $dz = \frac{v-az}{av}dv$, so that, replacing, we get

$$-\int\frac{dz}{\sqrt{a^2z^2-1}} = -\int\frac{v-az}{av}\times\frac{1}{v-az}dv$$

$$= -\frac{1}{a}\int\frac{dv}{v} = -\frac{1}{a}\log_\epsilon v,$$

so that $\int \frac{dy}{y\sqrt{a^2-y^2}} = -\frac{1}{a}\log_\epsilon \frac{a+\sqrt{a^2-y^2}}{y} + C_1.$

Now, for

$\int \frac{y\,dy}{\sqrt{a^2-y^2}}$ let $z=\sqrt{a^2-y^2}$; then $dz = -\frac{y\,dy}{\sqrt{a^2-y^2}}$;

hence

$$\int \frac{y\,dy}{\sqrt{a^2-y^2}} = -\int dz = -z = -\sqrt{a^2-y^2}+C_2.$$

We have then, finally,

$$x = a\log_\epsilon \frac{a+\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2}+C.$$

When $x=0$, $y=a$, so that $0=a\log_\epsilon 1 - 0 + C$, and $C=0$; the equation of the tractrix is therefore

$$x = a\log_\epsilon \frac{a+\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2}.$$

If $a=3$, as before, and if the length of the arc from $x=0$ to $x=1$ is required, it is not an easy matter to calculate the value of y corresponding to any given numerical value of x. It is, however, easy to find graphically an approximation as near the correct value as we desire, when we are given the value of a as follows:

Plot the graph, giving suitable values to y, say 3, 2, 1·5, 1. From this graph, find what values of y correspond to the two given values of x determining the arc, the length of which is needed, as accurately as the scale of the graph allows. For $x=0$, $y=3$ of course; suppose that for $x=1$ you find $y=1{\cdot}72$ on the graph. This is only approximate. Now plot again, on as large a scale as possible, taking only three values of y, 1·6, 1·7, 1·8. On this second graph, which is nearly, but not quite a straight line, you will be probably able to read any value of y correct to three places of decimals, and this is sufficient for our purpose. We find from the graph that $y=1{\cdot}723$ corresponds to $x=1$. Then

$$s=3\left[\log_\epsilon\frac{3}{y}\right]_{x=0}^{x=1}=3\left[\log_\epsilon\frac{3}{y}\right]_3^{1\cdot723}$$

$$=3(\log_\epsilon 1{\cdot}741-0)=1{\cdot}66.$$

If we wanted a more accurate value of y we could plot a third graph, taking for values of y 1·722, 1·723, 1·724, ...; this would give us, correct to five places of decimals, the value of y corresponding to $x=1$, and so on, till the required accuracy is reached.

Example 5. Find the length of an arc of the logarithmic spiral $r=\epsilon^\theta$ between $\theta=0$ and $\theta=1$ radian.

Do you remember differentiating $y=\epsilon^x$? It is an easy one to remember, for it remains always the same whatever is done to it: $\dfrac{dy}{dx}=\epsilon^x$ (see p. 143).

Here, since $\qquad r=\epsilon^\theta,\quad \dfrac{dr}{d\theta}=\epsilon^\theta=r.$

If we reverse the process and integrate $\int\epsilon^\theta\,d\theta$ we get back to $r+C$, the constant C being always introduced by such a process, as we have seen in Chap. XVII.

It follows that

$$s=\int\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta=\int\sqrt{(r^2+r^2)}\,d\theta$$

$$=\sqrt{2}\int r\,d\theta=\sqrt{2}\int\epsilon^\theta d\theta=\sqrt{2}\,(\epsilon^\theta+C).$$

Integrating between the two given values $\theta=0$ and $\theta=1$, we get

$$s=\int_0^1\sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta=\Big[\sqrt{2}(\epsilon^\theta+C)\Big]_0^1$$

$$=\sqrt{2}\epsilon^1-\sqrt{2}\epsilon^0=\sqrt{2}(\epsilon-1)$$

$$=1{\cdot}41\times1{\cdot}713=2{\cdot}42 \text{ inches},$$

since $r=\epsilon^0=1$ inch when $\theta=0$.

Example 6. Find the length of an arc of the logarithmic spiral $r=\epsilon^\theta$ between $\theta=0$ and $\theta=\theta_1$.

As we have just seen,

$$s=\sqrt{2}\int_0^{\theta_1}\epsilon^\theta\,d\theta=\sqrt{2}\Big[\epsilon^{\theta_1}-\epsilon^0\Big]=\sqrt{2}\left(\epsilon^{\theta_1}-1\right).$$

Example 7. As a last example let us work fully a case leading to a typical integration which will be found useful for several of the exercises found at the end of this chapter. Let us find the expression for the length of an arc of the curve $y=\frac{a}{2}x^2+3$.

$$\frac{dy}{dx}=ax,\quad s=\int\sqrt{1+a^2x^2}\,dx.$$

Integrate this by parts: let

$$u=\sqrt{1+a^2x^2} \quad \text{and} \quad dx=dv;$$

then $$x=v \quad \text{and} \quad du=\frac{a^2x\,dx}{\sqrt{1+a^2x^2}},$$

by the method of differentiation explained in Chap. IX.

Since $\int u\,dv=uv-\int v\,du$ (see p. 226), we have

$$\int\sqrt{1+a^2x^2}\,dx=x\sqrt{1+a^2x^2}-a^2\int\frac{x^2\,dx}{\sqrt{1+a^2x^2}}. \quad (1)$$

Also, we can write

$$\int\sqrt{1+a^2x^2}\,dx=\int\frac{(1+a^2x^2)\,dx}{\sqrt{1+a^2x^2}};$$

hence

$$\int\sqrt{1+a^2x^2}\,dx=\int\frac{dx}{\sqrt{1+a^2x^2}}+a^2\int\frac{x^2\,dx}{\sqrt{1+a^2x^2}}. \quad (2)$$

Adding (1) and (2) we get

$$2\int\sqrt{1+a^2x^2}\,dx=x\sqrt{1+a^2x^2}+\int\frac{dx}{\sqrt{1+a^2x^2}}. \quad (3)$$

Remains to integrate $\int\frac{dx}{\sqrt{1+a^2x^2}}$; for this purpose let $\sqrt{1+a^2x^2}=v-ax$; then

$$1+a^2x^2=v^2-2avx+a^2x^2 \quad \text{or} \quad 1=v^2-2avx.$$

Differentiating this, to get rid of the constant, we get

$$0=2v\,dv-2av\,dx-2ax\,dv \quad \text{or} \quad av\,dx=v\,dv-ax\,dv;$$

that is $dx=\frac{(v-ax)\,dv}{av}$; replacing in $\int\frac{dx}{\sqrt{1+a^2x^2}}$ we obtain

$$\int\frac{(v-ax)\,dv}{av\sqrt{1+a^2x^2}}=\frac{1}{a}\int\frac{(v-ax)\,dv}{v(v-ax)}=\frac{1}{a}\int\frac{dv}{v}=\frac{1}{a}\log_\epsilon v;$$

hence $$\frac{dx}{\sqrt{1+a^2x^2}}=\frac{1}{a}\log_\epsilon(ax+\sqrt{1+a^2x^2}).$$

Replacing in (3) and dividing by 2 we get, finally,

$$s=\int\sqrt{1+a^2x^2}\,dx$$
$$=\frac{x}{2}\sqrt{1+a^2x^2}+\frac{1}{2a}\log_\epsilon(ax+\sqrt{1+a^2x^2}),$$

which can easily be calculated between any given limits.

You ought now to be able to attempt with success the following exercises. You will find it interesting as well as instructive to plot the curves and verify your results by measurement where possible.

The integration is usually of the kind shown on p. 228, Ex. (5), or p. 229, Ex. (1), or p. 278, Ex. (7).

Exercises XXI. (For Answers, see p. 300.)

(1) Find the length of the line $y=3x+2$ between the two points for which $x=1$ and $x=4$.

(2) Find the length of the line $y=ax+b$ between the two points for which $x=a^2$ and $x=-1$.

(3) Find the length of the curve $y=\frac{2}{3}x^{\frac{3}{2}}$ between the two points for which $x=0$ and $x=1$.

(4) Find the length of the curve $y=x^2$ between the two points for which $x=0$ and $x=2$.

(5) Find the length of the curve $y=mx^2$ between the two points for which $x=0$ and $x=\frac{1}{2m}$.

(6) Find the length of the curves $r=a\cos\theta$ and $r=a\sin\theta$ between $\theta=\theta_1$ and $\theta=\theta_2$.

(7) Find the length of the curve $r=a\sec\theta$.

(8) Find the length of the arc of the curve $y^2=4ax$ between $x=0$ and $x=a$.

(9) Find the length of the arc of the curve

$$y=x\left(\frac{x}{2}-1\right)$$

between $x=0$ and $x=4$.

(10) Find the length of the arc of the curve $y=\epsilon^x$ between $x=0$ and $x=1$.

(*Note.* This curve is in rectangular coordinates, and is not the same curve as the logarithmic spiral $r=\epsilon^\theta$ which is in polar coordinates. The two equations are similar, but the curves are quite different.)

(11) A curve is such that the coordinates of a point on it are $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$, θ being a certain angle which varies between 0 and 2π. Find the length of the curve. (It is called a *cycloid*.)

(12) Find the length of an arc of the curve $y^2=mx$ between $x=0$ and $x=\frac{m}{4}$.

(13) Find the expression for the length of an arc of the curve $y^2=\frac{x^3}{a}$.

(14) Find the length of the curve $y^2=8x^3$ between the two points for which $x=1$ and $x=2$.

(15) Find the length of the curve $y^{\frac{2}{3}}+x^{\frac{2}{3}}=a^{\frac{2}{3}}$ between $x=0$ and $x=a$.

(16) Find the length of the curve $r=a(1-\cos\theta)$ between $\theta=0$ and $\theta=\pi$.

You have now been personally conducted over the frontiers into the enchanted land. And in order that you may have a handy reference to the principal results, the author, in bidding you farewell, begs to present you with a passport in the shape of a convenient collection of standard forms (see pp. 286, 287). In the middle column are set down a number of the functions which most commonly occur. The results of differentiating them are set down on the left; the results of integrating them are set down on the right. May you find them useful!

EPILOGUE AND APOLOGUE.

It may be confidently assumed that when this tractate "Calculus made Easy" falls into the hands of the professional mathematicians, they will (if not too lazy) rise up as one man, and damn it as being a thoroughly bad book. Of that there can be, from their point of view, no possible manner of doubt whatever. It commits several most grievous and deplorable errors.

First, it shows how ridiculously easy most of the operations of the calculus really are.

Secondly, it gives away so many trade secrets. By showing you that *what one fool can do, other fools can do also,* it lets you see that these mathematical swells, who pride themselves on having mastered such an awfully difficult subject as the calculus, have no such great reason to be puffed up. They like you to think how terribly difficult it is, and don't want that superstition to be rudely dissipated.

Thirdly, among the dreadful things they will say about "So Easy" is this: that there is an utter failure on the part of the author to demonstrate with rigid and satisfactory completeness the validity of sundry methods which he has presented in simple fashion, and has even *dared to use* in solving problems! But why should he not? You don't forbid the use of a watch to every person who does not know how to make one? You don't object to the musician playing on a violin that he has not himself constructed. You don't teach the rules of syntax to children until they have already become fluent in the *use* of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus.

One other thing will the professed mathematicians say about this thoroughly bad and vicious book: that the reason why it is *so easy* is because the author has left out all the things that are really difficult. And the ghastly fact about this accusation is that—*it is true!* That is, indeed, why the book has been written—written for the legion of innocents who have hitherto been deterred from acquiring the elements of the calculus by the stupid way in which its teaching is almost always presented. Any subject can be made repulsive by presenting it bristling with difficulties. The aim of this book is to enable beginners to learn its language, to acquire familiarity with its endearing simplicities, and to grasp its powerful methods of solving problems, without being compelled to toil through the intricate out-of-the-way (and mostly irrelevant) mathematical gymnastics so dear to the unpractical mathematician.

There are amongst young engineers a number on whose ears the adage that *what one fool can do, another can,* may fall with a familiar sound. They are earnestly requested not to give the author away, nor to tell the mathematicians what a fool he really is.

TABLE OF STANDARD FORMS

$\dfrac{dy}{dx}$ ←	— y —	→ $\int y\,dx$
	Algebraic.	
1	x	$\frac{1}{2}x^2+C$
0	a	$ax+C$
1	$x \pm a$	$\frac{1}{2}x^2 \pm ax+C$
a	ax	$\frac{1}{2}ax^2+C$
$2x$	x^2	$\frac{1}{3}x^3+C$
nx^{n-1}	x^n	$\dfrac{1}{n+1}x^{n+1}+C$
$-x^{-2}$	x^{-1}	$\log_\epsilon x+C$
$\dfrac{du}{dx}\pm\dfrac{dv}{dx}\pm\dfrac{dw}{dx}$	$u \pm v \pm w$	$\int u\,dx \pm \int v\,dx \pm \int w\,dx$
$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$	uv	No general form known
$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$	$\dfrac{u}{v}$	No general form known
$\dfrac{du}{dx}$	u	$ux-\int x\,du+C$
	Exponential and Logarithmic.	
ϵ^x	ϵ^x	ϵ^x+C
x^{-1}	$\log_\epsilon x$	$x(\log_\epsilon x-1)+C$
$0{\cdot}4343 \times x^{-1}$	$\log_{10} x$	$0{\cdot}4343x(\log_\epsilon x-1)+C$
$a^x \log_\epsilon a$	a^x	$\dfrac{a^x}{\log_\epsilon a}+C$
	Trigonometrical.	
$\cos x$	$\sin x$	$-\cos x+C$
$-\sin x$	$\cos x$	$\sin x+C$
$\sec^2 x$	$\tan x$	$-\log_\epsilon \cos x+C$
	Circular (Inverse).	
$\dfrac{1}{\sqrt{(1-x^2)}}$	$\text{arc}\sin x$	$x\cdot\text{arc}\sin x+\sqrt{1-x^2}+C$
$-\dfrac{1}{\sqrt{(1-x^2)}}$	$\text{arc}\cos x$	$x\cdot\text{arc}\cos x-\sqrt{1-x^2}+C$
$\dfrac{1}{1+x^2}$	$\text{arc}\tan x$	$x\cdot\text{arc}\tan x-\frac{1}{2}\log_\epsilon(1+x^2)+C$

$\dfrac{dy}{dx}$ ←	← y →	→ $\int y\,dx$
	Hyperbolic.	
$\cosh x$	$\sinh x$	$\cosh x + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$\operatorname{sech}^2 x$	$\tanh x$	$\log_\epsilon \cosh x + C$
	Miscellaneous.	
$-\dfrac{1}{(x+a)^2}$	$\dfrac{1}{x+a}$	$\log_\epsilon (x+a) + C$
$-\dfrac{x}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{1}{\sqrt{a^2+x^2}}$	$\log_\epsilon (x+\sqrt{a^2+x^2}) + C$
$\mp\dfrac{b}{(a \pm bx)^2}$	$\dfrac{1}{a \pm bx}$	$\pm\dfrac{1}{b}\log_\epsilon (a \pm bx) + C$
$\dfrac{-3a^2x}{(a^2+x^2)^{\frac{5}{2}}}$	$\dfrac{a^2}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{x}{\sqrt{a^2+x^2}} + C$
$a \cdot \cos ax$	$\sin ax$	$-\dfrac{1}{a}\cos ax + C$
$-a \cdot \sin ax$	$\cos ax$	$\dfrac{1}{a}\sin ax + C$
$a \cdot \sec^2 ax$	$\tan ax$	$-\dfrac{1}{a}\log_\epsilon \cos ax + C$
$\sin 2x$	$\sin^2 x$	$\dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$
$-\sin 2x$	$\cos^2 x$	$\dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$
$n \cdot \sin^{n-1} x \cdot \cos x$	$\sin^n x$	$-\dfrac{\cos x}{n}\sin^{n-1} x + \dfrac{n-1}{n}\int \sin^{n-2} x\,dx + C$
$-\dfrac{\cos x}{\sin^2 x}$	$\dfrac{1}{\sin x}$	$\log_\epsilon \tan \dfrac{x}{2} + C$
$-\dfrac{\sin 2x}{\sin^4 x}$	$\dfrac{1}{\sin^2 x}$	$-\operatorname{cotan} x + C$
$\dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x}$	$\dfrac{1}{\sin x \cdot \cos x}$	$\log_\epsilon \tan x + C$
$n \cdot \sin mx \cdot \cos nx + m \cdot \sin nx \cdot \cos mx$	$\sin mx \cdot \sin nx$	$\frac{1}{2}\cos(m-n)x - \frac{1}{2}\cos(m+n)x + C$
$2a \cdot \sin 2ax$	$\sin^2 ax$	$\dfrac{x}{2} - \dfrac{\sin 2ax}{4a} + C$
$-2a \cdot \sin 2ax$	$\cos^2 ax$	$\dfrac{x}{2} + \dfrac{\sin 2ax}{4a} + C$

ANSWERS.

Exercises I. (p. 25.)

(1) $\frac{dy}{dx}=13x^{12}$. (2) $\frac{dy}{dx}=-\frac{3}{2}x^{-\frac{5}{2}}$. (3) $\frac{dy}{dx}=2ax^{(2a-1)}$.

(4) $\frac{du}{dt}=2{\cdot}4t^{1{\cdot}4}$. (5) $\frac{dz}{du}=\frac{1}{3}u^{-\frac{2}{3}}$. (6) $\frac{dy}{dx}=-\frac{5}{3}x^{-\frac{8}{3}}$.

(7) $\frac{du}{dx}=-\frac{8}{5}x^{-\frac{13}{5}}$. (8) $\frac{dy}{dx}=2ax^{a-1}$.

(9) $\frac{dy}{dx}=\frac{3}{q}x^{\frac{3-q}{q}}$. (10) $\frac{dy}{dx}=-\frac{m}{n}x^{-\frac{m+n}{n}}$.

Exercises II. (p. 33.)

(1) $\frac{dy}{dx}=3ax^2$. (2) $\frac{dy}{dx}=13\times\frac{3}{2}x^{\frac{1}{2}}$. (3) $\frac{dy}{dx}=6x^{-\frac{1}{2}}$.

(4) $\frac{dy}{dx}=\frac{1}{2}c^{\frac{1}{2}}x^{-\frac{1}{2}}$. (5) $\frac{du}{dz}=\frac{an}{c}z^{n-1}$. (6) $\frac{dy}{dt}=2{\cdot}36t$.

(7) $\frac{dl_t}{dt}=0{\cdot}000012\times l_0$.

(8) $\frac{dC}{dV}=abV^{b-1}$, 0·98, 3·00 and 7·47 candle power per volt respectively.

(9) $\frac{dn}{dD}=-\frac{1}{LD^2}\sqrt{\frac{gT}{\pi\sigma}}$, $\frac{dn}{dL}=-\frac{1}{DL^2}\sqrt{\frac{gT}{\pi\sigma}}$,

$\frac{dn}{d\sigma}=-\frac{1}{2DL}\sqrt{\frac{gT}{\pi\sigma^3}}$, $\frac{dn}{dT}=\frac{1}{2DL}\sqrt{\frac{g}{\pi\sigma T}}$

(10) $\dfrac{\text{Rate of change of } P \text{ when } t \text{ varies}}{\text{Rate of change of } P \text{ when } D \text{ varies}} = -\dfrac{D}{t}.$

(11) 2π, $2\pi r$, πl, $\frac{2}{3}\pi rh$, $8\pi r$, $4\pi r^2$. (12) $\dfrac{dD}{dT} = \dfrac{0{\cdot}000012 l_t}{\pi}.$

Exercises III. (p. 46.)

(1) (*a*) $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dots$ (*b*) $2ax + b$. (*c*) $2x + 2a$.

(*d*) $3x^2 + 6ax + 3a^2$.

(2) $\dfrac{dw}{dt} = a - bt.$ (3) $\dfrac{dy}{dx} = 2x.$

(4) $14110x^4 - 65404x^3 - 2244x^2 + 8192x + 1379.$

(5) $\dfrac{dx}{dy} = 2y + 8.$ (6) $185{\cdot}9022654x^2 + 154{\cdot}36334.$

(7) $\dfrac{-5}{(3x+2)^2}.$ (8) $\dfrac{6x^4 + 6x^3 + 9x^2}{(1 + x + 2x^2)^2}.$

(9) $\dfrac{ad - bc}{(cx + d)^2}.$ (10) $\dfrac{anx^{-n-1} + bnx^{n-1} + 2nx^{-1}}{(x^{-n} + b)^2}.$

(11) $b + 2ct.$

(12) $R_0(a + 2bt)$, $R_0\left(a + \dfrac{b}{2\sqrt{t}}\right)$, $-\dfrac{R_0(a + 2bt)}{(1 + at + bt^2)^2}$ or $\dfrac{R^2(a + 2bt)}{R_0}.$

(13) $1{\cdot}4340(0{\cdot}000014t - 0{\cdot}001024)$, $-0{\cdot}00117$, $-0{\cdot}00107$, $-0{\cdot}00097$.

(14) $\dfrac{dE}{dl} = b + \dfrac{k}{i}$, $\dfrac{dE}{di} = -\dfrac{c + kl}{i^2}.$

Exercises IV. (p. 51.)

(1) $17 + 24x$; 24. (2) $\dfrac{x^2 + 2ax - a}{(x + a)^2}$; $\dfrac{2a(a+1)}{(x + a)^3}$

(3) $1 + x + \dfrac{x^2}{1 \times 2} + \dfrac{x^3}{1 \times 2 \times 3}$; $1 + x + \dfrac{x^2}{1 \times 2}.$

(4) (*Exercises III.*):

(1) (*a*) $\dfrac{d^2u}{dx^2} = \dfrac{d^3u}{dx^3} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$ (*b*) $2a$, 0.

(*c*) 2, 0. (*d*) $6x + 6a$, 6.

(2) $-b$, 0. (3) 2, 0.

(4) $56440x^3 - 196212x^2 - 4488x + 8192$.
$169320x^2 - 392424x - 4488$.

(5) 2, 0. (6) $371{\cdot}80453x$, $371{\cdot}80453$.

(7) $\frac{30}{(3x+2)^3}$, $-\frac{270}{(3x+2)^4}$.

(*Examples*, p. 41):

(1) $\frac{6a}{b^2}x$, $\frac{6a}{b^2}$. (2) $\frac{3a\sqrt{b}}{2\sqrt{x}} - \frac{6b\sqrt[3]{a}}{x^3}$, $\frac{18b\sqrt[3]{a}}{x^4} - \frac{3a\sqrt{b}}{4\sqrt{x^3}}$.

(3) $\frac{2}{\sqrt[3]{\theta^8}} - \frac{1{\cdot}056}{\sqrt[5]{\theta^{11}}}$, $\frac{2{\cdot}3232}{\sqrt[5]{\theta^{16}}} - \frac{16}{3\sqrt[3]{\theta^{11}}}$.

(4) $810t^4 - 648t^3 + 479{\cdot}52t^2 - 139{\cdot}968t + 26{\cdot}64$.
$3240t^3 - 1944t^2 + 959{\cdot}04t - 139{\cdot}968$.

(5) $12x + 2$, 12. (6) $6x^2 - 9x$, $12x - 9$.

(7) $\frac{3}{4}\left(\frac{1}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta^5}}\right) + \frac{1}{4}\left(\frac{15}{\sqrt{\theta^7}} - \frac{1}{\sqrt{\theta^3}}\right)$.

$\frac{3}{8}\left(\frac{1}{\sqrt{\theta^5}} - \frac{1}{\sqrt{\theta^3}}\right) - \frac{15}{8}\left(\frac{7}{\sqrt{\theta^9}} + \frac{1}{\sqrt{\theta^7}}\right)$.

Exercises V. (p. 64.)

(2) 64; 147·2; and 0·32 feet per second.

(3) $\dot{x} = a - gt$; $\ddot{x} = -g$. (4) 45·1 feet per second.

(5) 12·4 feet per second per second. Yes.

(6) Angular velocity = 11·2 radians per second; angular acceleration = 9·6 radians per second per second.

(7) $v = 20{\cdot}4t^2 - 10{\cdot}8$. $a = 40{\cdot}8t$. 172·8 in./sec., 122·4 in./sec².

(8) $v = \frac{1}{30\sqrt[3]{(t-125)^2}}$, $a = -\frac{1}{45\sqrt[3]{(t-125)^5}}$.

(9) $v = 0{\cdot}8 - \frac{8t}{(4+t^2)^2}$, $a = \frac{24t^2 - 32}{(4+t^2)^3}$, 0·7926 and 0·00211.

(10) $n = 2$, $n = 11$.

Exercises VI. (p. 73.)

(1) $\dfrac{x}{\sqrt{x^2+1}}$. (2) $\dfrac{x}{\sqrt{x^2+a^2}}$. (3) $-\dfrac{1}{2\sqrt{(a+x)^3}}$.

(4) $\dfrac{ax}{\sqrt{(a-x^2)^3}}$. (5) $\dfrac{2a^2-x^2}{x^3\sqrt{x^2-a^2}}$.

(6) $\dfrac{\frac{3}{2}x^2\left[\frac{8}{9}x(x^3+a)-(x^4+a)\right]}{(x^4+a)^{\frac{2}{3}}(x^3+a)^{\frac{3}{2}}}$. (7) $\dfrac{2a(x-a)}{(x+a)^3}$.

(8) $\frac{5}{2}y^3$. (9) $\dfrac{1}{(1-\theta)\sqrt{1-\theta^2}}$.

Exercises VII. (p. 75.)

(1) $\dfrac{dw}{dx}=-\dfrac{3x^2(3+3x^3)}{27\left(\frac{1}{2}x^3+\frac{1}{4}x^6\right)^3}$.

(2) $\dfrac{dv}{dx}=-\dfrac{12x}{\sqrt{1+\sqrt{2}+3x^2}\left(\sqrt{3}+4\sqrt{1+\sqrt{2}+3x^2}\right)^2}$.

(3) $\dfrac{du}{dx}=-\dfrac{x^2(\sqrt{3}+x^3)}{\sqrt{\left[1+\left(1+\frac{x^3}{\sqrt{3}}\right)^2\right]^3}}$.

Exercises VIII. (p. 91.)

(2) 1·44.

(4) $\dfrac{dy}{dx}=3x^2+3$; and the numerical values are:

3, $3\frac{3}{4}$, 6, and 15.

(5) $\pm\sqrt{2}$.

(6) $\dfrac{dy}{dx}=-\dfrac{4}{9}\dfrac{x}{y}$. Slope is zero where $x=0$; and is $\mp\dfrac{1}{3\sqrt{2}}$ where $x=1$.

(7) $m=4$, $n=-3$.

(8) Intersections at $x=1$, $x=-3$. Angles 153° 26′, 2° 28′.

(9) Intersection at $x=3·57$, $y=3·57$. Angle 16° 16′.

(10) $x=\frac{1}{3}$, $y=2\frac{1}{3}$, $b=-\frac{5}{3}$.

Exercises IX. (p. 109.)

(1) Min.: $x=0$, $y=0$; max.: $x=-2$, $y=-4$.

(2) $x=a$.

(4) $25\sqrt{3}$ square inches.

(5) $\frac{dy}{dx}=-\frac{10}{x^2}+\frac{10}{(8-x)^2}$; $x=4$; $y=5$.

(6) Max. for $x=-1$; min. for $x=1$.

(7) Join the middle points of the four sides.

(8) $r=\frac{2}{3}R$, $r=\frac{R}{2}$, no max.

(9) $r=R\sqrt{\frac{2}{3}}$, $r=\frac{R}{\sqrt{2}}$, $r=0{\cdot}8506\,R$.

(10) At the rate of $\frac{8}{r}$ square feet per second.

(11) $r=\frac{R\sqrt{8}}{3}$.

(12) $n=\sqrt{\frac{NR}{r}}$.

Exercises X. (p. 118.)

(1) Max.: $x=-2{\cdot}19$, $y=24{\cdot}19$; min.: $x=1{\cdot}52$, $y=-1{\cdot}38$.

(2) $\frac{dy}{dx}=\frac{b}{a}-2cx$; $\frac{d^2y}{dx^2}=-2c$; $x=\frac{b}{2ac}$ (a *maximum*).

(3) (*a*) One maximum and two minima.
(*b*) One maximum. ($x=0$; other points unreal.)

(4) Min.: $x=1{\cdot}71$, $y=6{\cdot}14$.

(5) Max: $x=-{\cdot}5$. $y=4$.

(6) Max.: $x=1{\cdot}414$, $y=1{\cdot}7675$.
Min.: $x=-1{\cdot}414$, $y=-1{\cdot}7675$.

(7) Max.: $x=-3{\cdot}565$, $y=2{\cdot}12$.
Min.: $x=+3{\cdot}565$, $y=7{\cdot}88$.

(8) $0{\cdot}4N$, $0{\cdot}6N$.

(9) $x=\sqrt{\frac{a}{c}}$.

(10) Speed 8·66 nautical miles per hour. Time taken 115·47 hours.
Minimum cost £112. 12*s*.

(11) Max. and min. for $x=7{\cdot}5$, $y=\pm 5{\cdot}414$. (See example no. 10, p. 72.)

(12) Min.: $x=\frac{1}{2}$, $y=0{\cdot}25$; max.: $x=-\frac{1}{3}$, $y=1{\cdot}408$.

Exercises XI. (p. 130.)

(1) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$. (2) $\dfrac{1}{x-1}+\dfrac{2}{x-2}$. (3) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$.

(4) $\dfrac{5}{x-4}-\dfrac{4}{x-3}$. (5) $\dfrac{19}{13(2x+3)}-\dfrac{22}{13(3x-2)}$.

(6) $\dfrac{2}{x-2}+\dfrac{4}{x-3}-\dfrac{5}{x-4}$.

(7) $\dfrac{1}{6(x-1)}+\dfrac{11}{15(x+2)}+\dfrac{1}{10(x-3)}$.

(8) $\dfrac{7}{9(3x+1)}+\dfrac{71}{63(3x-2)}-\dfrac{5}{7(2x+1)}$.

(9) $\dfrac{1}{3(x-1)}+\dfrac{2x+1}{3(x^2+x+1)}$. (10) $x+\dfrac{2}{3(x+1)}+\dfrac{1-2x}{3(x^2-x+1)}$.

(11) $\dfrac{3}{x+1}+\dfrac{2x+1}{x^2+x+1}$. (12) $\dfrac{1}{x-1}-\dfrac{1}{x-2}+\dfrac{2}{(x-2)^2}$.

(13) $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}+\dfrac{1}{2(x+1)^2}$.

(14) $\dfrac{4}{9(x-1)}-\dfrac{4}{9(x+2)}-\dfrac{1}{3(x+2)^2}$.

(15) $\dfrac{1}{x+2}-\dfrac{x-1}{x^2+x+1}-\dfrac{1}{(x^2+x+1)^2}$.

(16) $\dfrac{5}{x+4}-\dfrac{32}{(x+4)^2}+\dfrac{36}{(x+4)^3}$.

(17) $\dfrac{7}{9(3x-2)^2}+\dfrac{55}{9(3x-2)^3}+\dfrac{73}{9(3x-2)^4}$.

(18) $\dfrac{1}{6(x-2)}+\dfrac{1}{3(x-2)^2}-\dfrac{x}{6(x^2+2x+4)}$.

Exercises XII. (p. 153.)

(1) $ab(\epsilon^{ax}+\epsilon^{-ax})$. (2) $2at+\frac{2}{t}$. (3) $\log_\epsilon n$.

(5) npv^{n-1}. (6) $\frac{n}{x}$.

(7) $\frac{3\epsilon^{-\frac{x}{x-1}}}{(x-1)^2}$. (8) $6x\epsilon^{-5x}-5(3x^2+1)\epsilon^{-5x}$.

(9) $\frac{ax^{a-1}}{x^a+a}$. (10) $\frac{15x^2+12x\sqrt{x}-1}{2\sqrt{x}}$.

(11) $\frac{1-\log_\epsilon(x+3)}{(x+3)^2}$. (12) $a^x(ax^{a-1}+x^a\log_\epsilon a)$.

(14) Min.: $y=0{\cdot}7$ for $x=0{\cdot}694$.

(15) $\frac{1+x}{x}$. (16) $\frac{3}{x}(\log_\epsilon ax)^2$.

Exercises XIII. (p. 162.)

(1) Let $\frac{t}{T}=x$ ($\therefore t=8x$), and use the Table on page 159.

(2) $T=34{\cdot}627$; $159{\cdot}46$ minutes.

(3) Take $2t=x$; and use the Table on page 159.

(5) (*a*) $x^x(1+\log_\epsilon x)$; (*b*) $2x(\epsilon^x)^x$; (*c*) $\epsilon^{x^x}\times x^x(1+\log_\epsilon x)$.

(6) $0{\cdot}14$ second. (7) (*a*) $1{\cdot}642$; (*b*) $15{\cdot}58$.

(8) $\mu=0{\cdot}00037$, $31^m\frac{1}{4}$.

(9) i is $63{\cdot}4\%$ of i_0, $221{\cdot}55$ kilometers.

(10) Working as accurately as possible with a table of four-figure logarithms, $k=0{\cdot}1339$, $0{\cdot}1445$, $0{\cdot}1553$, mean $=0{\cdot}1446$; percentage errors :—$10{\cdot}2\%$, practically nil, $+71{\cdot}9\%$.

(11) Min. for $x=\frac{1}{\epsilon}$. (12) Max. for $x=\epsilon$.

(13) Min. for $x=\log_\epsilon a$.

Exercises XIV. (p. 173.)

(1) (i) $\frac{dy}{d\theta}=A\cos\left(\theta-\frac{\pi}{2}\right)$;

(ii) $\frac{dy}{d\theta}=2\sin\theta\cos\theta=\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$;

(iii) $\frac{dy}{d\theta}=3\sin^2\theta\cos\theta$ and $\frac{dy}{d\theta}=3\cos 3\theta$.

(2) $\theta=45^\circ$ or $\frac{\pi}{4}$ radians.

(3) $\frac{dy}{dt}=-n\sin 2\pi nt$.

(4) $a^x\log_\epsilon a\cos a^x$.

(5) $\frac{-\sin x}{\cos x}=-\tan x$.

(6) $18{\cdot}2\cos(x+26^\circ)$.

(7) The slope is $\frac{dy}{d\theta}=100\cos(\theta-15^\circ)$, which is a maximum when $(\theta-15^\circ)=0$, or $\theta=15^\circ$; the value of the slope being then $=100$. When $\theta=75^\circ$ the slope is $100\cos(75^\circ-15^\circ)=100\cos 60^\circ=100\times\frac{1}{2}=50$.

(8) $\cos\theta\sin 2\theta+2\cos 2\theta\sin\theta=2\sin\theta(\cos^2\theta+\cos 2\theta)$
$=2\sin\theta(3\cos^2\theta-1)$.

(9) $amn\theta^{n-1}\tan^{m-1}(\theta^n)\sec^2\theta^n$.

(10) $\epsilon^x(\sin^2 x+\sin 2x)$; $\epsilon^x(\sin^2 x+2\sin 2x+2\cos 2x)$.

(11) (i) $\frac{dy}{dx}=\frac{ab}{(x+b)^2}$; (ii) $\frac{a}{b}\epsilon^{-\frac{x}{b}}$; (iii) $\frac{1}{90^\circ}\times\frac{ab}{(b^2+x^2)}$.

(12) (i) $\frac{dy}{dx}=\sec x\tan x$;

(ii) $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$; (iii) $\frac{dy}{dx}=\frac{1}{1+x^2}$;

(iv) $\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}$; (v) $\frac{dy}{dx}=\frac{\sqrt{3}\sec x(3\sec^2 x-1)}{2}$.

(13) $\frac{dy}{d\theta}=4{\cdot}6(2\theta+3)^{1\cdot 3}\cos(2\theta+3)^{2\cdot 3}$.

(14) $\dfrac{dy}{d\theta}=3\theta^2+3\cos(\theta+3)-\log_\epsilon 3(\cos\theta\times 3^{\sin\theta}+3\theta)$.

(15) $\theta=\cot\theta$; $\theta=\pm 0{\cdot}86$; $y=\pm 0{\cdot}56$; is max. for $+\theta$, min. for $-\theta$.

Exercises XV. (p. 180.)

(1) $x^2-6x^2y-2y^2$; $\frac{1}{3}-2x^3-4xy$.

(2) $2xyz+y^2z+z^2y+2xy^2z^2$;
$2xyz+x^2z+xz^2+2x^2yz^2$;
$2xyz+x^2y+xy^2+2x^2y^2z$.

(3) $\dfrac{1}{r}\{(x-a)+(y-b)+(z-c)\}=\dfrac{(x+y+z)-(a+b+c)}{r}$. $\dfrac{2}{r}$.

(4) $dy=v\,u^{v-1}du+u^v\log_\epsilon u\,dv$.

(5) $dy=3\sin v\,u^2du+u^3\cos v\,dv$,
$dy=u\sin x^{u-1}\cos x\,dx+(\sin x)^u\log_\epsilon\sin x\,du$,
$dy=\dfrac{1}{v}\dfrac{1}{u}du-\log_\epsilon u\dfrac{1}{v^2}dv$.

(7) Minimum for $x=y=-\frac{1}{2}$.

(8) (*a*) Length 2 feet, width=depth=1 foot, vol.=2 cubic feet.

(*b*) Radius $=\dfrac{2}{\pi}$ feet $=7{\cdot}46$ in., length=2 feet, vol. $=2{\cdot}54$.

(9) All three parts equal; the product is maximum.

(10) Minimum $=\epsilon^2$ for $x=y=1$.

(11) Minimum $=2{\cdot}307$ for $x=\frac{1}{2}$, $y=2$.

(12) Angle at apex $=90°$; equal sides=length $=\sqrt[3]{2V}$.

Exercises XVI. (p. 190.)

(1) $1\frac{1}{3}$. (2) 0·6344. (3) 0·2624.

(4) (*a*) $y=\frac{1}{8}x^2+C$; (*b*) $y=\sin x+C$.

(5) $y=x^2+3x+C$.

Exercises XVII. (p. 205.)

(1) $\frac{4\sqrt{a}\,x^{\frac{3}{2}}}{3}+C.$ (2) $-\frac{1}{x^3}+C.$ (3) $\frac{x^4}{4a}+C.$

(4) $\frac{1}{3}x^3+ax+C.$ (5) $-2x^{-\frac{5}{2}}+C.$

(6) $x^4+x^3+x^2+x+C.$ (7) $\frac{ax^2}{4}+\frac{bx^3}{9}+\frac{cx^4}{16}+C.$

(8) $\frac{x^2+a}{x+a}=x-a+\frac{a^2+a}{x+a}$ by division. Therefore the answer is $\frac{x^2}{2}-ax+(a^2+a)\log_\epsilon(x+a)+C.$

(See pages 199 and 201.)

(9) $\frac{x^4}{4}+3x^3+\frac{27}{2}x^2+27x+C.$ (10) $\frac{x^3}{3}+\frac{2-a}{2}x^2-2ax+C.$

(11) $a^2(2x^{\frac{3}{2}}+\frac{9}{4}x^{\frac{4}{3}})+C.$ (12) $-\frac{1}{3}\cos\theta-\frac{1}{6}\theta+C.$

(13) $\frac{\theta}{2}+\frac{\sin 2a\theta}{4a}+C.$ (14) $\frac{\theta}{2}-\frac{\sin 2\theta}{4}+C.$

(15) $\frac{\theta}{2}-\frac{\sin 2a\theta}{4a}+C.$ (16) $\frac{1}{3}\epsilon^{3x}+C.$

(17) $\log_\epsilon(1+x)+C.$ (18) $-\log_\epsilon(1-x)+C.$

Exercises XVIII. (p. 224.)

(1) Area $=60$; mean ordinate $=10$.

(2) Area $=\frac{2}{3}$ of $a\times 2a\sqrt{a}$.

(3) Area $=2$; mean ordinate $=\frac{2}{\pi}=0·637$.

(4) Area $=1·57$; mean ordinate $=0·5$.

(5) 0·572, 0·0476. (6) Volume $=\pi r^2\frac{h}{3}$.

(7) 1·25. (8) 79·6.

(9) Volume $=4·9348$; (from 0 to π).

(10) $a\log_\epsilon a$, $\frac{a}{a-1}\log_\epsilon a$.

(12) Arithmetical mean $=9·5$; quadratic mean $=10·85$.

(13) Quadratic mean $=\frac{1}{\sqrt{2}}\sqrt{A_1^2+A_3^2}$; arithmetical mean $=0$.

The first involves a somewhat difficult integral, and may be stated thus: By definition the quadratic mean will be

$$\sqrt{\frac{1}{2\pi}\int_0^{2\pi}(A_1\sin x+A_3\sin 3x)^2\,dx.}$$

Now the integration indicated by

$$\int(A_1^2\sin^2 x+2A_1A_3\sin x\sin 3x+A_3^2\sin^2 3x)\,dx$$

is more readily obtained if for $\sin^2 x$ we write

$$\frac{1-\cos 2x}{2}.$$

For $2\sin x\sin 3x$ we write $\cos 2x-\cos 4x$; and, for $\sin^2 3x$,

$$\frac{1-\cos 6x}{2}.$$

Making these substitutions, and integrating, we get (see p. 202)

$$\frac{A_1^2}{2}\left(x-\frac{\sin 2x}{2}\right)+A_1A_3\left(\frac{\sin 2x}{2}-\frac{\sin 4x}{4}\right)+\frac{A_3^2}{2}\left(x-\frac{\sin 6x}{6}\right).$$

At the lower limit the substitution of 0 for x causes all this to vanish, whilst at the upper limit the substitution of 2π for x gives $A_1^2\pi+A_3^2\pi$. And hence the answer follows.

(14) Area is 62·6 square units. Mean ordinate is 10·42.

(16) 436·3. (This solid is pear shaped.)

Exercises XIX. (p. 233.)

(1) $\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$

(2) $\frac{x^2}{2}(\log_\epsilon x-\frac{1}{2})+C.$

(3) $\frac{x^{a+1}}{a+1}\left(\log_\epsilon x-\frac{1}{a+1}\right)+C.$

(4) $\sin\epsilon^x+C.$

(5) $\sin(\log_\epsilon x)+C.$

(6) $\epsilon^x(x^2-2x+2)+C.$

(7) $\frac{1}{a+1}(\log_\epsilon x)^{a+1}+C.$ (8) $\log_\epsilon(\log_\epsilon x)+C.$

(9) $2\log_\epsilon(x-1)+3\log_\epsilon(x+2)+C.$

(10) $\frac{1}{2}\log_\epsilon(x-1)+\frac{1}{5}\log_\epsilon(x-2)+\frac{3}{10}\log_\epsilon(x+3)+C.$

(11) $\frac{b}{2a}\log_\epsilon\frac{x-a}{x+a}+C.$ (12) $\log_\epsilon\frac{x^2-1}{x^2+1}+C.$

(13) $\frac{1}{4}\log_\epsilon\frac{1+x}{1-x}+\frac{1}{2}\arctan x+C.$

(14) $\frac{1}{\sqrt{a}}\log_\epsilon\frac{\sqrt{a}-\sqrt{a-bx^2}}{x\sqrt{a}}.$ (Let $\frac{1}{x}=v$; then, in the result, let $\sqrt{v^2-\frac{b}{a}}=v-u.$)

You had better differentiate now the answer and work back to the given expression as a check.

Exercises XX. (p. 263.)

(1) $r=2\sqrt{2},\ x_1=-2,\ y_1=3.$ (2) $r=2\cdot83,\ x_1=0,\ y_1=2.$

(3) $x=\pm0\cdot383,\ y=0\cdot147.$ (4) $r=2,\ x_1=y_1=2\sqrt{m}.$

(5) $r=2a,\ x_1=2a+3x,\ y_1=-\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ when $x=0,\ x_1=2a,\ y_1=0.$

(6) When $x=0,\ r=y_1=$ infinity, $x_1=0.$
When $x=+0\cdot9,\ r=3\cdot36,\ x_1=-2\cdot21,\ y_1=+2\cdot01.$
When $x=-0\cdot9,\ r=3\cdot36,\ x_1=+2\cdot21,\ y=-2\cdot01.$

(7) When $x=0,\ r=1\cdot41,\ x_1=1,\ y_1=3.$
When $x=1,\ r=1\cdot41,\ x_1=0,\ y_1=3.$
Minimum $=1\cdot75.$

(8) For $x=-2,\ r=112\cdot3,\ x_1=109\cdot8,\ y_1=-17\cdot2.$
For $x=0,\ r=x_1=y_1=$ infinity.
For $x=1,\ r=1\cdot86,\ x_1=-0\cdot67,\ y_1=-0\cdot17.$

(9) $x=-0\cdot33,\ y=+1\cdot08.$

(10) $r=1,\ x=2,\ y=0$ for all points. A circle.

(11) When $x=0,\ r=1\cdot86,\ x_1=1\cdot67,\ y_1=0\cdot17.$
When $x=1\cdot5,\ r=0\cdot365,\ x_1=1\cdot59,\ y_1=0\cdot98.$
$x=1,\ y=1$ for zero curvature.

(12) When $\theta=\frac{\pi}{2}$, $r=1$, $x_1=\frac{\pi}{2}$, $y_1=0$.

When $\theta=\frac{\pi}{4}$, $r=2{\cdot}598$, $x_1=2{\cdot}285$, $y_1=-1{\cdot}41$.

(14) When $\theta=0$, $r=1$, $x_1=0$, $y_1=0$.

When $\theta=\frac{\pi}{4}$, $r=2{\cdot}598$, $x_1=0{\cdot}7146$, $y_1=-1{\cdot}41$.

When $\theta=\frac{\pi}{2}$, $r=x_1=y_1=$ infinity.

(15) $r=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, where $x=0$, $r=\frac{a^2}{b}$, $x_1=0$, $y_1=\frac{b^2-a^2}{b}$.

Exercise XXI. (p. 280.)

(1) $s=9{\cdot}48$. (2) $s=(1+a^2)^{\frac{3}{2}}$ (3) $s=1{\cdot}21$.

(4) $s=\int_0^2 \sqrt{1+4x^2}\,dx=\left[\frac{x}{2}\sqrt{1+4x^2}+\frac{1}{4}\log_\epsilon(2x+\sqrt{1+4x^2})\right]_0^2$
$=4{\cdot}64$.

(5) $s=\frac{0{\cdot}57}{m}$. (6) $s=a(\theta_2-\theta_1)$. (7) $s=\sqrt{r^2-a^2}$.

(8) $s=\int_0^a\sqrt{1+\frac{a}{x}}\,dx$ and $s=a\sqrt{2}+a\log_\epsilon(1+\sqrt{2})$.

(9) $s=\frac{x-1}{2}\sqrt{(x-1)^2+1}+\frac{1}{2}\log_\epsilon\{(x-1)+\sqrt{(x-1)^2+1}\}$ and
$s=6{\cdot}80$.

(10) $s=\int\frac{dy}{y\sqrt{1+y^2}}+\int\frac{y\,dy}{\sqrt{1+y^2}}$. Let $y=\frac{1}{z}$ in the first and $\sqrt{1+y^2}=v^2$ in the second; this leads to $s=\sqrt{1+y^2}+\log_\epsilon\frac{y}{1+\sqrt{1+y^2}}$ and $s=2{\cdot}00$.

(11) $s=2a\int\sin\frac{\theta}{2}\,d\theta$ and $s=8a$.

(12) $s=\sqrt{x}\sqrt{x+\frac{m}{4}}+\frac{m}{4}\log_\epsilon\left(\sqrt{x}+\sqrt{x+\frac{m}{4}}\right)$ and

$$s=\frac{m}{4}\sqrt{2}+\frac{m}{4}\log_\epsilon(1+\sqrt{2}).$$

(13) $s=\frac{8a}{27}\left\{1+\left(\frac{9x}{4a}\right)\right\}^{\frac{3}{2}}.$

(14) $s=\int_1^2\sqrt{1+18x}\,dx.$ Let $1+18x=z$, express s in terms of z and integrate between the values of z corresponding to $x=1$ and $x=2$. $s=5{\cdot}27.$

(15) $s=\frac{3a}{2}.$ (16) $4a.$

Every earnest student is exhorted to manufacture more examples for himself at every stage, so as to test his powers. When integrating he can always test his answer by differentiating it, to see whether he gets back the expression from which he started.

There are lots of books which give examples for practice. It will suffice here to name two: R. G. Blaine's *The Calculus and its Applications*, and F. M. Saxelby's *A Course in Practical Mathematics*.